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Edinburgh Patrons of the Univers

as the Reward of his Proficienc

in the Second Mathematical

Claſs during the Seſsion 1821-2

William Wallace

Profr. of Mathemat

(I.C.)

A
TREATISE
ON
PLANE AND SPHERICAL
Trigonometry.

By ROBERT WOODHOUSE, A.M. F.R.S.
FELLOW OF GONVILLE AND CAIUS COLLEGE.

THE THIRD EDITION,
Corrected, Altered, and Enlarged.

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1819

PREFACE

TO THE SECOND EDITION.

It is not easy to adapt a Treatise on Trigonometry to all descriptions of Students; to state, in its beginning, within a small compass, and with their simplest solutions, those Propositions which relate merely to the cases of oblique-angled triangles, and then, on the ground of those propositions and by the method of their solutions, to proceed to investigations of greater intricacy.

The Student, if he be supposed to possess a knowledge of the first six Books of Euclid, may thence, by a few easy inferences, and by the aid of some simple constructions, arrive most readily at the Trigonometrical solutions of the cases of oblique-angled triangles. If his views extend no farther, he cannot take a better guide than Ludlam or Robert Simson; nor proceed by any easier method than the *Geometrical.*

But few Students are content with such confined views. Trigonometry is now extended far beyond its original object, and to other investigations than those of the relations of the sides and angles of Triangles. The *collateral* uses of the science have become the most numerous, and are not the least important. To the knowledge of many of these, however, the Geometrical method is unable to conduct us. At some point or other of our enquiries (we speak of its present and actual state) it must be abandoned, and recourse be had to that which technically is called the *Analytical Method.*

Since this latter is the sole thoroughly efficient method, will it not be better to make it, in a Treatise on Trigonometry, the predominant one, instead of being compelled merely to call in its aid, when the resources of the former are exhausted?

The Author of the present Treatise has endeavoured to construct it on such a plan; and, with this view, he has had as little recourse as possible, to Geometrical constructions and the properties of figures. What he thence has borrowed are not so much to be considered as the first steps in his process of demonstration, as the data and ground-work from which the process itself is to commence and to be instituted.

By these means the process is made uniform and systematical. But uniformity may be purchased at too dear a rate; and the main purpose of the Work, which is utility, would be sacrificed, if, for the sake of system, the analytical method were reluctantly compelled to submit to modes of proof that are strange to its nature and genius.

The specimens of demonstration contained in the following pages must determine whether or not such sacrifice has been made.

The great practical use of Trigonometry is the resolution of rectilinear triangles; but, that it is capable of being extended, and to objects, not merely curious, but of real interest, we may learn from the history and actual state of the science.

The first considerable extension of Trigonometry, beyond its original object, was made about twenty years after the death of

Newton. It was then, on the ground-work laid down by that great man, that three Mathematicians of the Continent, Clairaut, Dalembert and Euler, and Thomas Simpson our countryman, began to establish a system of Physical Astronomy more perfect than what its Author had left. With this view, they laid aside the Geometrical method which Newton had used, and which they thought incompetent to their purpose, and adopted the Analytical. Pursuing this method, they perceived the formulæ of Trigonometry to be of continual use and recurrence, and the language, by which the process of demonstration was conducted, to be formed, in a great degree, of symbols and phrases borrowed from that science. In order, therefore, to render the process precise and expeditious, it became necessary to improve the means and instruments by which it was carried on; and, accordingly, at the time spoken of, the advancement of Trigonometry, the pure and subsidiary science, was contemporaneous with that of Astronomy, the mixed and principal one.

This general statement would be confirmed by an examination of the Memoirs and Treatises on Physical Astronomy published about the year 1750.

Clairaut and Dalembert in their *Lunar Theories* embody in those Works, or introduce as prefatory matter, several, now commonly known, Trigonometrical formulæ *. In the Volume of Tracts which Thomas Simpson published, the Author evidently intended the one which is inserted at p. 76, as preparatory to the succeeding Theory of the Moon; and Euler distinctly states as a reason for cultivating the algorithm of sines, its great utility in the mixed Mathematics.

---

* It will hardly be believed that theorems, such as are given in pp. 28, 29, &c. were almost unknown. Yet Clairaut, (*Mem. Acad.* 1745, p. 342, and *Theorie de la Lune,* edit. 2, p. 9.) alluding to these Theorems, says, 'M. Euler est le premier, que je sçache, qui ait fait usage de ces Theoremes pour operer sur les sinus et cosinus d'angles, sans avoir recours a leurs formes imaginaires.'

In the arrangement of the Treatise, which the Table of Contents sufficiently explains, Spherical succeeds to plane Trigonometry. Now, the former has not, like the latter, been extended beyond its original purpose. It has no collateral and indirect uses; it has not enriched the general language of analysis, by its peculiar phrases. But, notwithstanding this confined range, and apparent simplicity in the object of the science, its propositions are more easily established by the Analytical method than the Geometrical. And, (at least in the opinion of the Author of this Treatise) this would be the case, even if there existed no similarity and artificial connexion, between the processes by which the series of formulæ in the two branches of Trigonometry were respectively established. But, so far from there being no similarity, the corresponding propositions can be deduced by methods so analogous, that to know the one is almost to know the other.

This will appear to be the case, if we refer to pages 24 and 139, &c. of this Treatise. We shall there find similar Algebraical derivations of formulæ from two fundamental expressions for the cosine of an angle. The principle of the derivation, however, is not new; it originated with Euler, who inserted in the *Acta Acad. Petrop.* for 1779, a Memoir entitled *Trigonometria Spherica Universa, ex primis principiis breviter et dilucide derivata.* Gua next, in the *Memoirs of the Academy of Sciences* for 1783, p. 291, deduced, but by awkward and complicated processes, Spherical Trigonometry "from the Algebraical solution of the simplest of its Problems." In 1786, Cagnoli, in his excellent Treatise, derived without "similar triangles or complicated figures," the fundamental expressions for the sine and cosine of the sum of two arcs. And lastly, Lagrange and Legendre, the one in the *Journal de L'Ecole Polytechnique*, the other in his *Elemens de Geometrie*, have followed and simplified Euler's method, and instead of three fundamental expressions, have shewn one to be sufficient.

PREFACE

TO THE THIRD EDITION.

THE present Edition is not different, in its plan, from the second. It contains, however, more matter, to the amount of twenty-five pages, and some alterations. The additional matter (the greater portion of it) will be found in the enquiry concerning the properties and uses of logarithms; the altered matter at the beginning of the Treatise. But whatever is changed or added, has been so with a view of rendering the Work more plain and easy to the Student; and, for the same end, it has been interspersed with new instances of illustration.

Caius College, Feb. 2, 1819.

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&\sin.(A \pm B) = \sin. A . \cos. B \pm \cos. A . \sin. B\\
&\cos.(A \pm B) = \cos. A . \cos. B \mp \sin. A . \sin. B\\
&\sin.(A + B) + \sin.(A - B) = 2 \sin. A . \cos. B\\
&\sin.(A + B) - \sin.(A - B) = 2 \cos. A . \sin. B\\
&\sin.(A + B) . \sin.(A - B) = \sin.^2 A - \sin.^2 B\\
&\cos.(A - B) + \cos.(A + B) = 2 \cos. A . \cos. B\\
&\cos.(A - B) - \cos.(A + B) = 2 \sin. A . \sin. B\\
&\sin. A \pm \sin. B = 2 \sin. \frac{A \pm B}{2} . \cos. \frac{A \mp B}{2}\\
&\cos. A + \cos. B = 2 \cos. \frac{A + B}{2} . \cos. \frac{A - B}{2}
\end{aligned}\right\} 27 - 33$$

Page

$$\left.\begin{array}{l}
\cos. B - \cos. A = 2 \sin. \frac{A+B}{2} \cdot \sin. \frac{A-B}{2} \\
\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\tan. \frac{A+B}{2}}{\tan. \frac{A-B}{2}} \ldots\ldots\ldots\ldots \\
\frac{\cos. A + \cos. B}{\cos. B - \cos. A} = \frac{1}{\tan. \frac{A+B}{2} \cdot \tan. \frac{A-B}{2}} \ldots\ldots \\
\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan. \frac{A+B}{2} \ldots\ldots\ldots\ldots \\
\frac{\sin. A}{1 + \cos. A} = \tan. \frac{A}{2} \ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\} 33$$

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$$\left.\begin{array}{l}
\sin. 2A = 2 \sin. A \cos. A \ldots\ldots\ldots\ldots \\
\sin. 3A = 3 \sin. A - 4 \sin.^3 A \ldots\ldots\ldots\ldots \\
\sin. 4A = (4 \sin. A - 8 \sin.^3 A) \cos. A \ldots\ldots \\
\sin. 5A = 5 \sin. A - 20 \sin.^3 A + 16 \sin.^5 A \ldots
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$2^5 \cos.^6 A = \cos. 6A + 6 . \cos. 4A + 15 \cos. 2A + 10$

$2 \sin.^2 A = - \cos. 2A + 1$
$2^2 \sin.^3 A = - \sin. 3A + 3 \sin. A$
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PLANE TRIGONOMETRY.

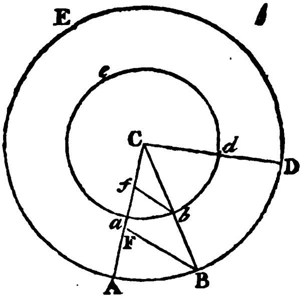
CHAP. I.

On the Division of the Circle into Degrees, Minutes, Seconds, &c.—Definitions of Sines, Cosines, &c.

ARCS THE MEASURES OF ANGLES.

ART. 1. IT is proved in the 33d Proposition of the sixth Book of Euclid, that, in equal circles, angles have the same ratio to each other, as the arcs on which they stand. Hence also, in the same circle, the angles vary as the arcs on which they stand; and consequently we may assume arcs as the measures of angles.

In the circle $ABDE$, the arcs AB, AD, are measures of the angles ACB, ACD; and of the same angles, in the smaller circle $abde$, ab, ad, are the measures; which latter arcs have the same ratio to the arcs AB, AD, that the radius Ca has to CA. For, since, in the circle $abde$, the measure of four right angles is the whole circumference $abde$,



$$\angle\, aCb : 4 \text{ right } \angle s :: ab : abde, \text{ therefore}$$

$$\angle\, aCb = 4 \text{ right } \angle s \times \frac{ab}{abde}.$$

Similarly, $\angle\, ACB$, or $\angle\, aCb = 4$ right $\angle s \times \dfrac{AB}{ABDE}$.

Hence, $\dfrac{ab}{AB} = \dfrac{abde}{ABDE} = \dfrac{Ca}{CA}$, since, (Playfair's *Geometry*, edit. 2. p. 219.) the circumferences of circles are to one another as their radii.

2. If from the points B, b, two lines BF, bf be drawn making with CA equal angles CFB, Cfb, then, by the similar triangles CFB, Cfb, we have

$$\frac{BF}{bf} = \frac{CB}{Cb},$$

and hence, if, in a circle $ABDE$, we have determined the value of a line such as BF, we can always assign the value of a similar line bf, in another circle $abde$, provided the ratio of the radii CB, Cb, be known: for instance, if CB be called 1; and Cb, r,

$$bf = BF \times \frac{r}{1} = BF \times r, \text{ or } BF = \frac{bf}{r}.$$

3. * It is usual to divide the circumference of a circle into 360 equal parts, which parts are called degrees, and of which the symbol is n^0 or 5^0, if n or 5 be their number: each degree is also divided into 60 equal parts, which parts are called minutes; and of which the symbol is m' or $7'$, if m or 7 be their number: and, finally, each minute is divided into 60 equal parts, which parts are called seconds, and of which the symbol is t'' or $35''$, if t or 35 be their number: thus, if AB (Fig. p. 1.) equals one-fourth of the circle $ABDE$, AB contains 90 degrees, or, symbolically, $AB = 90^0$. If $AB = \frac{1}{17}$th of the circumference $ABDE$,

---

* See the Note at the end of the Chapter.

AB contains 30° or $AB = 30^\circ$. If $AB = \frac{1}{7}$th of the circumference

$$ABDE,\ AB = \tfrac{1}{7}\,360^\circ = 51^\circ + \frac{3}{7}\,1^\circ = 51^\circ + \frac{3.60'}{7}$$

$$= 51^\circ\,25' + \frac{5}{7}\,1' = 51^\circ\,25'\,42'' + \frac{6}{7}\,1''.$$

The values of $\frac{6}{7}\,1''$ and of like quantities are, usually, expressed by means of decimals; thus, $\frac{6}{7}\,1''$, retaining only the two first figures, equals $0''.85$; and $\frac{1}{7}$th of the circumference would be expressed by

$$51^\circ\,25'\,42''.85.$$

But it is occasionally useful (see *Astron.* p. 397.) to extend the division of the circle beyond that of seconds, and to introduce, with their proper symbols, *thirds*, *fourths*, &c. In such an extension, $\frac{6}{7}\,1''$ would equal $51'''\,25^{\text{iv}}.7$ and the foregoing arc, the seventh of the circumference, would be expressed by

$$51^\circ\,25'\,42''\,51'''\,25^{\text{iv}}.7.$$

4. The arcs of circles, it has appeared, are proper measures of the angles which they subtend; if the angles be increased, the arcs are also increased, and in the same ratio; and knowing the value of one, is, in fact, knowing the value of the other. But, in Trigonometry, the values of angles are made to depend on the values of certain right lines, drawn according to certain rules, but not varying as the angles vary. The lines just alluded to are called sines, tangents, secants, &c. which it now becomes necessary to define.

The *Sine* of an Arc is a right line drawn, from one extremity of an arc, perpendicularly to a diameter passing through the other extremity.

The *Cosine* of an Arc is a right line intercepted between the centre of the circle and that point in the diameter (the *foot* of the sine) at which the sine of the same arc drawn perpendicularly to the diameter meets it.

The *Versed Sine* is a part of the diameter intercepted between the common extremity of the arc and diameter, and the *foot* of the sine.

The *Tangent* of an Arc is a right line, drawn from one extremity of it and perpendicularly to a diameter passing through it, and terminated by its intersection with another diameter passing through the other extremity of the arc and produced beyond it.

The *Secant* of an Arc is a line intercepted between the centre of the circle and that extremity of the tangent of the same arc which lies without the circle.

The *Chord* of an Arc is a straight line joining its two extremities.

The *Complement* of an Arc less than a quadrant is its defect from a quadrant. The *Co-tangent*, *Co-secant* of an arc are, respectively, the tangent, and secant of its complement, and, therefore, may be drawn according to the preceding directions, by considering the *complement* of the arc to be the arc itself.

If we now illustrate these definitions, and assume, in the annexed diagram, AB to be the arc: then, see p. 3. l. 24.

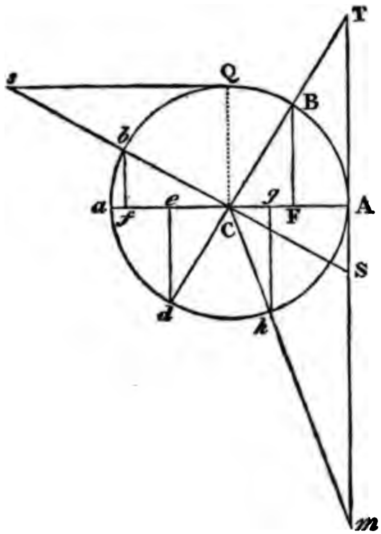
BF is the *sine* of AB, and F is what we have called the *foot* of the sine.

CF is the *cosine* of AB (see p. 3. l. 27.)
AF is the *versed sine* of AB (l. 3.)
AT is the *tangent* of AB (l. 6.)
CT is the secant of AB.

A line joining A and B would be the chord of AB: as Bgb, bfb' (see fig. p. 7.) are the chords of the arcs BQb, bab'.

The complement of AB is (l. 15.) BQ, since $AB+BQ=$quadrant: now *BQ* being considered as the arc of which the sine, tangent, secant, &c. are required, its sine, cosine, tangent, secant, by the preceding definitions, are (see fig. p. 7.) respectively, *Bg*, *Cg*, *Qt*, *Ct*: and, accordingly, (see pp. 3, 4.) those same lines are respectively, the cosine, sine, co-tangent, co-secant of the arc *AB*.

The lines that have hitherto been drawn expound the sine, cosine, &c. of an arc *AB* less than a quadrant: but if we take



Ab greater than a quadrant, then, according to the above definitions, *bf* is the sine of the arc *AQb*.

Cf is the cosine,
Af is the versed sine,
CS the secant,
AS is the tangent.

In order to determine the co-tangent and co-secant of this arc *Ab* we must vary and extend the definition of the complement of an arc: now, (see p. 4. l. 15.) the arc being less than a quadrant, its complement was defined to be its *defect* from that quantity: but an extended definition which should make the *complement* of an arc to be the difference between it and a quadrant would suit both arcs greater and less than a quadrant; and, according to such definition, Qb (fig. p. 5.) would be the *complement* of Ab, and Qs (AQ being a quadrant) the tangent of Qb would be the *co-tangent* of Ab. Cs the secant of Qb, would be the *co-secant* of Ab.

Let the arc be called A, then when A is less than a quadrant (Q),

$$A + (Q - A) = Q,$$
$$\text{or, } A - (A - Q) = Q,$$

when A is greater than a quadrant (Q),

$$A - (A - Q) = Q,$$

therefore, by what has preceded,

$$\text{sin. } A = \text{cos. } (A - Q), \text{ and sin. } (A - Q) = \text{cos. } A,$$
$$\text{tan. } A = \text{co-tan. } (A - Q), \text{ and co-tan. } A = \text{tan. } (A - Q),$$
$$\text{sec. } A = \text{co-sec. } (A - Q), \text{ and co-sec. } A = \text{sec. } (A - Q).$$

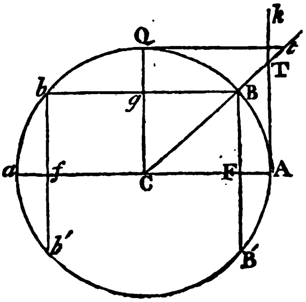
Let now the arc be greater than two quadrants but less than three: and let $AQad$ represent such an arc, then, by the preceding definitions (fig. p. 5.),

de is the sine,
Ce the cosine,
Ae the versed sine,
AT the tangent,
CT the secant.

Lastly, let the arc be greater than three quadrants but less than four, or less than the circumference of the circle; and let $AQak$ represent such an arc, then

kg is its sine,
Cg the cosine,
Ag the versed sine,
Am the tangent,
Cm the secant.

We may also use the definitions of p. 3, 4, and draw the sines, cosines, &c. of arcs *greater* (if we may so call them) than the circumference: for instance, suppose the arc to be equal to the circumference plus the arc AB; then, guided by the definition (see p. 3,) begin from A, one extremity, and go round the circle in the direction $AQaB'$ till you arrive at B the other extremity of the arc ($= AQaA + AB$): from B draw BF perpendicularly to Aa passing through A, and BF is the required sine of the arc, CF is the cosine, AF the versed sine, AT the tangent, and CT the secant: which are evidently the sine, cosine, &c. of the arc AB. Hence, admitting the existence of arcs greater than a circle, and applying to such the original definitions of p. 3, 4, we have these equalities



$$\text{sin. } AB = \text{sin. (circumference} + AB),$$

$$\text{or sin. } A = \text{sin. } (2\pi + A),$$

calling AB, A, and the circumference 2π; also

$$\text{cos. } A = \text{cos. } (2\pi + A),$$
$$\text{tan. } A = \text{tan. } (2\pi + A),$$
$$\text{sec. } A = \text{sec. } (2\pi + A),$$

and in a similar manner we might easily obtain more like equalities.

The sine, cosine of an arc are thus expressed by means of the sine, cosine, &c. of other arcs: but they may also be expressed in terms of one another: thus, since by the forty-seventh Proposition of the first Book of Euclid,

$$CB^2 = CF^2 + BF^2,$$

we have, making $r = CB$,

$$r^2 = \cos^2. A + \sin.^2 A,$$

and, accordingly,

$$\sin.^2 A = r^2 - \cos.^2 A.$$

Again, since $CT^2 = CA^2 + AT^2$,

$$\sec.^2 A = r^2 + \tan.^2 A,$$

and, accordingly,

$$\tan.^2 A = \sec.^2 A - r^2.$$

Again, by the similar triangles CFB, CAT,

$$CF : FB :: CA : AT;$$

$$\therefore AT = CA \times \frac{FB}{CF},$$

$$\text{or, } \tan. A = r . \frac{\sin. A}{\cos. A}.$$

and, by the same similar triangles,

$$CT = CB \times \frac{CA}{CF}$$

$$= \text{rad.} \times \frac{\text{rad.}}{\cos. AB}$$

$$= \frac{r^2}{\cos. A}.$$

In like manner we may easily deduce from the similar triangles CQt, CgB, the values of Qt and Ct, the *co-tangent* and *co-secant* (see p. 6.) of the arc AB, $(= A)$,

$$\text{for, } Qt = CQ \times \frac{Bg}{Cg} = \text{rad.} \times \frac{\sin. QB}{\cos. QB}$$

$$= \text{rad.} \frac{\cos. AB}{\sin. AB}$$

$$= \frac{r . \cos. A}{\sin. A},$$

and,

$$Ct = CB \times \frac{CQ}{Cg} = \text{rad.} \times \frac{\text{rad.}}{\sin. AB}$$

$$= \frac{r^2}{\sin. A}.$$

Or, we may dispense with this second set of similar triangles (CQt and CgB) and deduce the values of the co-tangent and co-secant from the previous values of the tangent and secant, by means of their definition (see p. 6.)

Thus, the co-tangent of an arc is the tangent of the complement of that arc. If A be the arc and Q a quadrant, $Q-A$ is the complement: now by l. 19. of p. 8.

$$\tan. (Q - A) = r . \frac{\sin. (Q - A)}{\cos. (Q - A)};$$

$$\text{but, } \sin. (Q - A) = \cos. A,$$

$$\cos. (Q - A) = \sin. A;$$

$$\therefore \text{ co-tan. } A \text{ [the same as tan. } (Q - A)] = r . \frac{\cos. A}{\sin. A}.$$

In like manner,

$$\text{co-sec. } A = \text{sec. } (Q - A) = \frac{r^2}{\cos. (Q - A)}$$

$$= \frac{r^2}{\sin. A}.$$

If it were worth the while it would be easy to express, under different terms, the preceding equalities: for instance, we may thus express the two latter.

The radius is a mean proportional to the secant and cosine of an arc, and, also, to the co-secant and sine.

The radius is also a mean proportional to the tangent and co-tangent: which may be thus deduced,

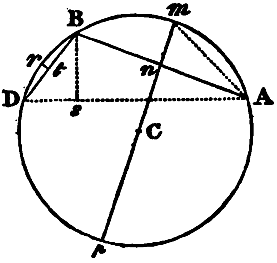
$$\text{tan.}\,A = r.\frac{\text{sin.}\,A}{\text{cos.}\,A},\ \text{co-tan.}\,A = r.\frac{\text{cos.}\,A}{\text{sin.}\,A};$$

$$\therefore\ \text{tan.}\,A \times \text{co-tan.}\,A = r^2,$$

$$\text{or, tan.}\,A : r :: r : \text{co-tan.}\,A,$$

and this same proportion is immediately deducible from the similar triangles CQt, CAT (fig. p. 7.).

The right line AB is (see p. 4.) the *chord* of the arc AB. If the right line be bisected at n and Cn be drawn perpendicularly to AB, then the point m, where Cn produced meets the circle, bisects the arc AB (Euclid, Book III, Prop. XXX.); therefore



$$\text{the arc } Am = Bm = \frac{AB}{2} = \frac{A}{2} \text{ (supposing } A = AB).$$

But An is by the definition of p. 3. l. 24, the sine of Am. the chord (AnB), therefore, of an arc A is double the sine (An) of half the arc $\left(\frac{A}{2}\right)$: or, which is the same proposition, the sine of an arc (A) is half the chord of twice that arc ($2A$).

Instead of making A = the arc AB, make, for convenience, $2A$ to represent it, and let v represent mn the versed sine of Am or Bm ($= A$): then, since (Euclid, Book III. Prop. 35.),

$$m\,n \times n\,p = A\,n^2,$$

$$v \times (2\,r - v) = \sin.^2 A,$$

$$\text{or, } 2\,r\,v = \sin.^2 A + v^2.$$

But $A\,n^2 + m\,n^2 = A\,m^2$ (the line $A\,m$),

$$\text{or, } \sin.^2 A + v^2 = \left(2 \sin. \frac{A}{2}\right)^2 \text{ (see p. 10. ll. 17, \&c.);}$$

$$\therefore \text{ (see l. 6.) } 2\,r\,v = 4 . \sin.^2 \frac{A}{2};$$

$$\text{but } C\,n = C\,m - m\,n,$$

$$\text{or, } \cos. \frac{A}{2} = r - v$$

$$= \text{, therefore, } r - \frac{2}{r} \sin.^2 \frac{A}{2}$$

$$= \frac{1}{r}\left(r^2 - 2 \sin^2. \frac{A}{2}\right),$$

$$\text{or, } = \frac{1}{r}\left(2 \cos.^2 \frac{A}{2} - r^2\right)$$

since

$$\cos.^2 \frac{A}{2} + \sin.^2 \frac{A}{2} = r^2.$$

If we multiply the two last expressions (ll. 13, 14.) for cos. A, we have

$$\cos. A \times \cos. A, \text{ or } \cos.^2 A =$$

$$\frac{1}{r^2}\left\{2\,r^2\left(\cos.^2 \frac{A}{2} + \sin.^2 \frac{A}{2}\right) - r^4 - 4 \sin.^2 \frac{A}{2} \cos.^2 \frac{A}{2}\right\}$$

$$= \frac{1}{r^2}\left(r^4 - 4 . \sin.^2 \frac{A}{2} \cos.^2 \frac{A}{2}\right).$$

Hence, $\frac{4}{r^2} \sin.^2 \frac{A}{2} \cos.^2 \frac{A}{2} = r^2 - \cos.^2 A$

$$= \sin.^2 A,$$

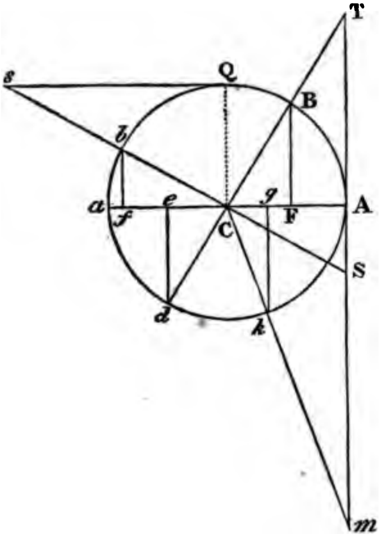
and consequently,

$$2 \sin. \frac{A}{2} \cos. \frac{A}{2} = r \sin. A.$$

The *supplement* of an arc is the difference between it and a semicircle: accordingly,

BQa is the *supplement* of *AB*,
ab is the *supplement* of *AQb*,
and *ab* is the *supplement* of *AQad*.

Now (see the definition) *BF* is the sine of *BQa*: for *BF* is drawn from one extremity *B* of the arc *BQa* perpendicularly to the diameter *aA* passing through *a* the other extremity: but, as we have seen (p. 4.), or by the same definition, *BF* is the sine of *AB*. Similarly, *bf* is the sine equally of the arc *ba* and the arc *AQb*: *ed* the sine of *ad* and *aQd*: and, generally, *the sine of the supplement of an arc is equal the sine of the arc itself.*



⁂ We have already (see pp. 6, 7.) in one or two instances, expressed by means of general symbols certain equalities that subsist between the sines and cosines of arcs, which, though different, may be said to be related. Such modes of expression are, in the actual

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⁂ The parts included within ⁂ ⁂ are less essential than the other parts, and may, in a first or partial perusal, be passed over by the Student.

operations, carried on by the aid of the Trigonometrical Analysis, of frequent and considerable use. It is desirable, therefore, to extend them, which may be done without much difficulty. Thus with regard to the equality which has just been stated: if 2π[†] represent the circumference of a circle of which the radius is 1, and A be the arc, then

$$\sin. A = \sin. (\pi - A),$$

and since A may represent any arc, if instead of A we substitute $\frac{\pi}{2} - A$ in the preceding equation, we have

$$\sin. \left(\frac{\pi}{2} - A\right) = \sin. \left(\frac{\pi}{2} + A\right),$$

and if, instead of A, we substitute

$\frac{m}{n}\pi - A$, we have

$$\sin. \left(\frac{m}{n}\pi - A\right) = \sin. \left(\frac{n-m}{n}\pi + A\right).$$

We may also obtain other general expressions; thus by extending the definition of a sine (see pp. 4, 7.) the subjoined equations are true

$$\sin. A = \sin. (2\pi + A) = \sin. (4\pi + A), \text{ and generally,}$$

$$\sin. A = \sin. (2n\pi + A),$$

n being a whole number.

For like reasons,

$$\sin. (\pi - A) = \sin. (3\pi - A) = \sin. (5\pi - A),$$

and generally,

$$\sin. (\pi - A) = \sin. \{(2n + 1)\pi - A\},$$

n being any number in the progression, 0, 1, 2, 3, &c. Hence,

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[*] The numerical value of π, that of the circumference of a circle of which the *diameter* is 1, is 3.14159, &c.: 2π, or 2×3.14159, &c. expresses, then, the value of the circumference of a circle, of which the radius is 1.

since by p. 12, $\sin. A = \sin. (\pi - A)$ there results this general equation,

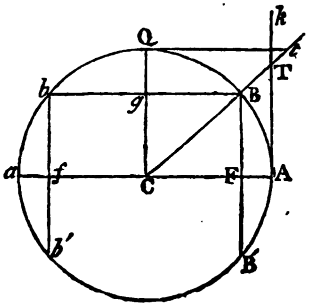
$$\sin. (2n\pi + A) = \sin. \{ (2n + 1)\pi - A \},$$

or, what amounts to the same thing, if s be the sine of any arc A, it is also the sine of all arcs comprehended under the two formulæ,

$$(2n\pi + A), \quad \{ (2n + 1)\pi - A \},$$

in which n may be any term of the progression, 0, 1, 2, 3, &c.

11. The definition of a cosine being (see p. 7.) like that of a sine, extended to designate the cosines of arcs, that are greater than the circumference, we may, in like manner, obtain general expressions for it. Thus, CF, which is the cosine of AB, may be considered as the cosine of the (circumference + AB) &c. Hence, as before the following equations will be true; $\cos. A = \cos. (2\pi + A) = \cos. (4\pi + A) = \cos. (2n\pi + A)$, n being any term of the progression, 0, 1, 2, 3, 4, 5, &c. But, since the same CF is also the cosine of the arc $ABaB'$, we have $CF = \cos. (2\pi - A) =$ (for reasons just alledged) $\cos. (4\pi - A) = \cos. (2n\pi - A)$, n being any term of the progression, 1, 2, 3, &c. or, CF, generally, =



$$\cos. \{ (2n + 2)\pi - A \},$$

n being any term of the progression, 0, 1, 2, 3, &c.

Hence CF is the cosine of all arcs comprehended within the two formulæ

$$(2n\pi + A), \quad \{ (2n + 2)\pi - A \}.$$

12. In a similar manner we may investigate the general formulæ of arcs that have the same *negative* cosine,

$Cf = \cos.(\pi - A) = \cos.(3\pi - A) =$, generally, $\cos.\ \{(2n+1)\pi - A\}$, n being any term of the progression, 0, 1, 2, 3, &c.
and since the same Cf is the cosine also of $ABbab'$ or $\pi + A$

$$Cf = \cos.(\pi + A) = \cos.(3\pi + A) = \cos.\ \{(2n+1)\pi + A\}.$$

Hence, Cf is the cosine of all arcs comprehended within the two formulæ,

$$\{(2n+1)\pi - A\},\quad \{(2n+1)\pi + A\}.$$

The arcs of which fb' is the sine, are

$$\pi + A,\ 3\pi + A, \text{ and generally } \{(2n+1)\pi + A\}.$$

The arcs of which $FB' = fb'$ is the sine, are

$$2\pi - A,\ 4\pi - A, \text{ and generally } (2n+2)\pi - A.$$

Hence FB' is the sine of all arcs comprehended within the two formulæ,

$$\{(2n+1)\pi + A\} \text{ and } \{(2n+2)\pi - A\},$$

n being in each case any term of the progression, 0, 1, 2, 3, &c.

In calculations, where FB, FB', and other quantities are involved, if s be the symbol for FB, $-s$ must be the symbol for FB'. For, conceive a line to be drawn a tangent to the circle, at the point opposite to Q in QC produced, and let the distance of any point in the circunference from this line be called z, then $FB\ (s) = z - r$, and $FB' = r - z$, or $FB' = -(z - r)$. Hence, if in any equation subsisting between trigonometrical lines we wish to pass from the consideration of the point B to that of the point B', we must in such equation substitute $-(z-r)$ instead of $z - r$, or $-s$, instead of s.*

13. The preceding results may be conveniently represented in a Table, s and c representing the sine and cosine of an arc A.

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* This hinges on the general doctrine of negative quantities: the scrupulous Student, who is not satisfied with what is here said, is referred to Carnot's *Geometrie de position,* and his subsequent work on the Theory of Transversals, &c.

Values of	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.	Arcs.
n	sine $= s$	sine $= s$	sine $= -s$	sine $= -s$	cos. $= c$	cos. $= c$	cos. $= -c$	cos. $= -c$
0	A	$\pi - A$	$\pi + A$	$2\pi - A$	A	$2\pi - A$	$\pi - A$	$\pi + A$
1	$2\pi + A$	$3\pi - A$	$3\pi + A$	$4\pi - A$	$2\pi + A$	$4\pi - A$	$3\pi - A$	$3\pi + A$
2	$4\pi + A$	$5\pi - A$	$5\pi + A$	$6\pi - A$	$4\pi + A$	$6\pi - A$	$5\pi - A$	$5\pi + A$
3	$6\pi + A$	$7\pi - A$	$7\pi + A$	$8\pi - A$	$6\pi + A$	$8\pi - A$	$7\pi - A$	$7\pi + A$
4	$8\pi + A$	$9\pi - A$	$9\pi + A$	$10\pi - A$	$8\pi + A$	$10\pi - A$	$9\pi - A$	$9\pi + A$
&c.								
General Formulæ.	$2n\pi + A$	$(2n+1)\pi - A$	$(2n+1)\pi + A$	$(2n+2)\pi - A$	$2n\pi + A$	$(2n+2)\pi - A$	$(2n+1)\pi - A$	$(2n+1)\pi + A$

It is easy from this Table and the expressions for tan. A, co-tan. A, sec. A, &c. namely,

$$\frac{\sin. A}{\cos. A},\quad \frac{\cos. A}{\sin. A},\quad \frac{1}{\cos. A}\ (\text{rad.} = 1)$$

to determine the values of the tangent, co-tangent, secant, &c.

$$\text{thus } \tan. (\pi + A) = \frac{\sin. \pi + A)}{\cos. (\pi + A)} = \frac{-s}{-c}$$

$$= \tan. A,$$

$$\text{co-tan.} (3\pi - A) = \frac{\cos. (3\pi - A)}{\sin. (3\pi - A)} = \frac{-c}{s}$$

$$= - \text{co-tan.} A,$$

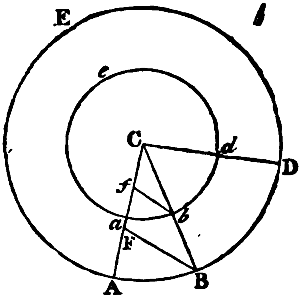
$$\sec. (7\pi + A) = \frac{1}{\cos. (7\pi + A)} = \frac{1}{-c} = - \sec. A.$$

18. In some of the preceding expressions, a radius $= 1$ has been used, and, solely, for the purpose of lessening the number of symbols in the Trigonometrical formulæ. For, 1, or any power or root of it, used as a multiplier or divisor of an expression, may be expunged from such expression; thus, instead of $\frac{\sin.^2 A}{1^2}$, we may more simply write $\sin.^2 A$. Still, however, it is, on many occasions, necessary to use, for the radius, a general symbol such as r, or, an arithmetical value such as 10,000. For this reason, it is desirable to be possessed of some easy and expeditious rule, for converting formulæ constructed with a radius $= 1$ into other formulæ that shall have a different radius. Such a rule may be obtained from the following simple considerations:

If (see fig. of next page) BF, bf be drawn similarly inclined to CA, then by similar triangles,

$$BF = bf . \frac{CB}{Cb}, \text{ or } BF = bf . \frac{1}{r} \text{ (if } CB = 1,\ Cb = r).$$

In like manner, $CF = Cf . \frac{1}{r}$, $AF = af \frac{1}{r}$. Now, sines, cosines, tangents, &c. are drawn after the manner that these lines are; if the angle BFC be a right angle, BF is the sine of the angle BCF to radius 1, bf is the sine to radius r; and, CF, Cf, are the cosines. Hence, in any formula involving sin. A, cos. A, &c.



calculated for a radius = 1, substitute instead of sin. A, cos. A; $\frac{\sin. A}{r}$, $\frac{\cos. A}{r}$, and the resulting formula will belong to lines drawn in a circle, of which the radius is r: for instance,

$$\tan. A = \frac{\sin. A}{\cos. A}, \text{ radius} = 1; \therefore \frac{\tan. A}{r} = \frac{\sin. A}{r} \times \frac{r}{\cos. A},$$

$$\text{or, } \tan. A = r . \frac{\sin. A}{\cos. A}, \text{ (radius } = r).$$

$$\text{Again,} \quad \sec. A = \frac{1}{\cos. A} \text{ (radius } = 1), \text{ then}$$

$$\frac{\sec. A}{r} = \frac{1}{\frac{\cos. A}{r}} = \frac{r}{\cos. A}, \text{ or}$$

$$\sec. A = \frac{r^2}{\cos. A}, \text{ (radius} = r).$$

$$\text{Again, } \tan. A . \text{co-tan.} A = 1, \text{ then, } \frac{\tan. A}{r} . \frac{\text{co-tan.} A}{r} = 1,$$

$$\text{and } \tan. A . \text{co-tan.} A = r^2.$$

And if in any formula, any power of sin. A, or of cos. A, such as $\sin.^3 A$, $\sin.^n A$, $\cos.^4 A$, or $\cos.^n A$, occurs, the radius being 1, then, by substituting $\frac{\sin.^3 A}{r^3}$, $\frac{\sin.^n A}{r^n}$, $\frac{\cos.^4 A}{r^4}$, $\frac{\cos.^n A}{r^n}$, the resulting formula will belong to a circle of which the radius is r.

This is the rule; but, since it would leave a Trigonometrical formula with fractional terms (the denominators being powers of the radius) it may, with advantage, be modified and made more convenient. Thus, suppose a form should occur such as

$$\cos. nA = a \cdot \cos.^n A + b \cdot \cos.^{n-2} A + \&c. \text{ to radius } 1;$$

then, by what had preceded, the reduced form to a radius r, is

$$\frac{\cos. nA}{r} = \frac{a . \cos.^n A}{r^n} + b \cdot \frac{\cos.^{n-2} A}{r^{n-2}} + \&c.$$

and, cleared of fractions, is

$$r^{n-1} \cos. nA = a \cdot \cos.^n A + b \cdot r^2 . \cos.^{n-2} A + \&c.$$

Here, $\cos. nA$ is multiplied by r^{n-1}, $\cos.^{n-2} A$ by r^2, &c.; that is, if we choose to call $\cos.^n A$ a quantity of n dimensions, $\cos.^{n-2} A$, a quantity of $n - 2$ dimensions, $\cos. A \times \sin. A$, a quantity of two dimensions, we may announce the preceding rule under the following simple form:

Multiply each term of a Trigonometrical formula, in which the radius $= 1$, *by such power of* r, *as shall make it of the same dimensions with the term of the highest dimensions; the resulting formula will be true when the radius is* $=$ r.

Thus, if

$$\cos. 3A = 4 . \cos.^3 A - 3 . \cos. A \qquad (\text{rad}^s. = 1),$$

since $\cos.^3 A$, the term of the highest, is of three dimensions, and $\cos. 3A$, $\cos. A$, are of one dimension, we have

$$r^2 . \cos. 3A = 4 . \cos.^3 A - 3r^2 \cos. A.$$

19. The Trigonometrical symbols, such as $\sin. A$, $\tan. A$, &c. that have been obtained, are merely general ones, and, hitherto, no methods have been given of assigning their values in specific values of the angles. The *general* methods for this purpose will be given in a subsequent part of the Treatise; but, even at this stage, by peculiar artifices, we may, in certain simple cases, assign the arithmetical values of the sines and cosines of angles. For example,

If (fig. p. 14.) $AB = BQ$, that is, if AB = half a quadrant, or expressed in degrees, if AB or $\angle ABC = 45^0$, since $\angle BCQ = 45^0$, also cos. $45^0 =$ sin. 45^0, but cos.$^2 A +$ sin.$^2 A = 1$, (1 = radius in this case); therefore, $2(\text{sin. } 45^0)^2 = 1$, and sin. $45^0 = \frac{1}{\sqrt{2}} = .7071068$, or, (see the preceding rule) $= 7071.068$, to a radius $= 10{,}000$.

If $ACB = \frac{1}{3} 90^0 = 30^0$, since $BCB' = 2 . ACB = 60^0$, and since $\angle CBB' = \angle CB'B = 60^0$, the triangle BCB' is equilateral, and consequently BB' (the chord of 60^0) = radius $CA = 1$, and, $BF =$ sin. $30^0 = \frac{1}{2} BB' = \frac{1}{2} = .5$, and $\therefore$ cos. 30^0 or sin. 60^0 (see Art. 10.) $= \sqrt{\left(1 - \frac{1}{4}\right)} = \frac{\sqrt{3}}{2} = .8660254$, and (see p. 18.) =

8660.254 to a radius = 10,000,
and 8660254 to a radius = 1000000.

Hence may be proved, what was asserted in p. 3, that the sines of arcs do not vary as the arcs themselves. For, the sin. $30^0 = \frac{1}{2}$ radius $=, \therefore, \frac{1}{2}$ sin. 90^0; in other words, the sines are as 1 to 2, whilst the arcs are as 1 to 3.

The values of the tangent may be found, in the above cases, from the expression tan. $A = \frac{\text{sin. } A}{\text{cos. } A}$. Thus

$$\text{rad}^s = 1. \qquad\qquad \text{rad}^s. = 10{,}000$$

$$\text{tan. } 45^0 = \frac{\text{sin. } 45^0}{\text{cos. } 45^0} = 1 \ldots\ldots\ldots\ldots = 10{,}000$$

$$\text{tan. } 30^0 = \frac{1}{2} \times \frac{2}{\sqrt{3}} = .5773503 \ldots\ldots = 5773.503$$

$$\text{tan. } 60^0 = \frac{\sqrt{3}}{2} \times \frac{2}{1} = 1.7320508 \ldots\ldots = 17320.508$$

$$\text{tan. } 90^0 = \frac{\text{sin. } 90^0}{\text{cos. } 90^0} = \frac{1}{0} = \infty$$

We may here direct the Student's attention to the superior augmentation of the tangent above that of the sine; for, we have corresponding to

* Arcs	0,	30°	45°	60°	90°,
Sines	0,	5000,	7071,	8660,	10000,
Tangents	0,	5773,	10000,	17320,	∞.

In the same manner as we have found, in the preceding cases, the values of the tangents, we may find those of the versed sine, secant, &c.

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* We have adhered, in this Chapter, to the antient and common division of the circle. But, in most of the French scientific treatises that have, of late years, been published, the circumference is divided first, into 400 equal parts or degrees, then, each degree into 100 equal parts, or minutes, then, each minute into 100 equal parts or seconds: so that a French degree is less than an English in the proportion of 90 to 100: a French minute less than an English, in the proportion of 90×60 to 100×100: and a French second less in the proportion of $90 \times 60 \times 60$ to $100 \times 100 \times 100$: hence, if n be the number of French degrees, the corresponding number of English equals $\frac{n.9}{10}$, or $\frac{n(10-1)}{10}$, or $n-\frac{n}{10}$, which last form points to an easy arithmetical operation for finding the number of degrees in the English scale from the number in the French scale: since from the proposed number we must subtract the same, after the decimal point has been moved one place to the left:

Examples: What number of degrees, minutes, &c. in the English scale correspond to 73°, to 71° 15′, and to 26°.0735, in the French scale.

73	71.15	26.0735
7.3	7.115	2.60735
65.7	64.035	23.46615
6	6	6
65° 42′ English.	2.10	27 . 9690
	6	6
	6	58.140 Answer 23° 27′ 58″.
	64° 2′ 6″	

In the next Chapter we will proceed to investigate certain expressions for the sine and cosine of the sum and difference of two arcs, in terms of the sines and cosines of the simple arcs. Such expressions are, in this science, very important, since from them, by an easy derivation, may be made to flow almost all other Trigonometrical formulæ. ⁂

This operation of reducing French to English degrees may be superseded, and rendered less liable to mistake, by means of the opposite Table, in which, as it is usual, the reduction is effected simply by addition.

Example to the Table.

Reduce 26°.0735 to English degrees, &c.

	French.			English.		
By the Table,	20°	0′	0″	18°	0′	0″
	6	0	0	5	24	0
	0	.07	0	0	3	46.8
	0	0	30	0	0	9.72
	0	0	5	0	0	1.62
	26 . 0735			23	27	58.14 the same as before.

Reduce 2°.7483 to English degrees, &c.

	French.			English.		
By Table,	2°	0′	0″	1°	48′	0″
	0	.70	0	0	37	48
	0	4	0	0	2	9.6
	0	0	80	0	0	25.92
	0	0	3	0	0	.972.
	2° . 7483			2	28	24.492

TABLE

For Reducing French Degrees, &c. to English.

French Division :

400° in the circle, 100′, in a degree, 100″ in a minute.

English Division :

360°.................60′..................60″.

	French.	English.	French.	English.
Degrees.	1°	0° 54′	10°	9°
	2	1 48	20	18
	3	2 42	30	27
	4	3 36	40	36
	5	4 30	50	45
	6	5 24	60	54
	7	6 18	70	63
	8	7 12	80	72
	9	8 6	90	81
	10	9 0	100	90
Minutes.	1′	0′ 32″.4	10	5′ 24″
	2	1 4.8	20	10 48
	3	1 37.2	30	16 12
	4	2 9.6	40	21 36
	5	2 42	50	27 0
	6	3 14.4	60	32 24
	7	3 46.8	70	37 48
	8	4 19.2	80	43 12
	9	4 51.6	90	48 36
	10	5 24	100	54 0
Seconds.	1″	0″.324	10	3″.24
	2	0.648	20	6.48
	3	0.972	30	9.72
	4	1.296	40	12.96
	5	1.62	50	16.2
	6	1.944	60	19.44
	7	2.268	70	22.68
	8	2.592	80	25.92
	9	2.916	90	29.16
	10	3.24	100	32.4

CHAP. II.

Expressions, for the Sines and Cosines of the Angles of a Triangle, in Terms of the Sides: for the Sine and Cosine of the Sum and Difference of two Arcs or Angles, &c.

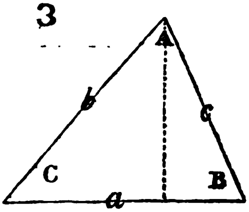
THE first step in this investigation will be made by the solution of the following Problem:

PROBLEM 1. In an oblique-angled triangle, it is required to express the cosines of the angles in terms of the sides.

Let the 3 angles be A, B, C, the opposite sides, a, b, c.

Let the line between the vertex of the angle C and the point where a perpendicular from the vertex of A on the line a cuts a be p; then, p is called the cosine of C to the radius b, and (by p. 17.) $= b . \cos. C$, when the radius $= 1$.

3



Now, by *Euclid*, Book II, Prop. 12 and 13,

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ap \\ &= a^2 + b^2 - 2ab . \cos. C; \end{aligned}$$

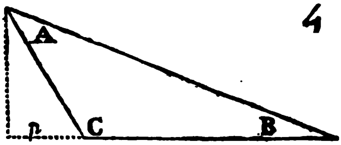
$$\text{consequently, } \cos. C = \frac{a^2 + b^2 - c^2}{2ab}.$$

If we investigate cos. B, and cos. A, the process will be exactly similar, and the result similar, that is

$$\cos. B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc}.$$

From these expressions, the angles of a triangle may be found when the sides are given.



If C be obtuse, then $p = b . \cos. (\pi - C)$,

$$\text{and } c^2 = a^2 + b^2 + 2ap$$
$$= a^2 + b^2 + 2ab . \cos. (\pi - C)$$
$$= a^2 + b^2 - 2ab . \cos. C \quad \{ \text{since } \cos. (\pi - C) = -\cos. C \};$$

consequently, $\cos. C = \frac{a^2 + b^2 - c^2}{2ab}$, as before, or the expression is the same, whether C be less or greater than a right angle.

PROBLEM 2. Let it be required to express the sines of the angles in terms of the sides.

By *Euclid*, Book I. Prop. 47. $\sin.^2 A = (\text{rad}^s.)^2 - \cos.^2 A$ $=$ (when $\text{rad}^s. = 1$) $1 - \cos.^2 A = (1 + \cos. A)(1 - \cos. A)$; since the difference of the squares of two quantities is equal to the product of their sum and difference; hence we may find the value of $\sin.^2 A$, by finding, from the preceding Problem, $1 + \cos. A$, and $1 - \cos. A$, and, then, by multiplying together those quantities.

$$\text{Now, } 1 - \cos. A = 1 - \frac{b^2 + c^2 - a^2}{2bc}.$$
$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc}$$
$$= \frac{a^2 - (b - c)^2}{2bc}$$
$$= \frac{(a + b - c)(a + c - b)}{2bc}, \text{ (by I. 14, \&c.)}$$

similarly, [*] $1 + \cos. A = \frac{(b^2 + 2bc + c^2) - a^2}{2bc}$

$$= \frac{(b + c)^2 - a^2}{2bc}$$

$$= \frac{(a+b+c)(b+c-a)}{2bc}$$ (By p. 25. l. 14, &c.)

Hence multiplying, $1 + \cos. A$, and $1 - \cos. A$,

$$\sin.^2 A = \frac{(a+b+c)(b+c-a)(a+b-c)(a+c-b)}{4b^2c^2};$$

or, since $a+b-c = 2\left(\frac{a+b+c}{2} - c\right)$, $b+c-a = 2\left(\frac{a+b+c}{2} - a\right)$, &c. $\sin.^2 A =$

$$\frac{2.2.2.2}{4b^2c^2}\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - b\right) \times \left(\frac{a+b+c}{2} - c\right)\right\} =$$

$$\frac{2^2}{b^2c^2}\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - b\right)\left(\frac{a+b+c}{2} - c\right)\right\},$$

This is the expression for the $\sin.^2 A$, formed by means of that for cos. A. But, the expressions for cos. B, cos. C are precisely similar to that for cos. A, and, therefore, the $\sin.^2 B$ and $\sin.^2 C$ formed from them, by the same process, must be expressed by similar fractions; in which fractions, the numerators must, from the nature of their composition, be the same as the numerator for $\sin.^2 A$, and the denominators will be, respectively, a^2c^2, a^2b^2. Let N^2 represent the numerator, then

$$\sin. A = \frac{N}{bc}, \ \sin. B = \frac{N}{ac}, \ \sin. C = \frac{N}{ab}.$$

---

[*] $1 + \cos. A = \text{ver. sin.}\ (180^\circ - A)$;

$\therefore$ ver. sin. $(180^\circ - A) = \frac{(b + c)^2 - a^2}{2bc}$, or, differently expressed,

$4bc : (b + c)^2 - a^2 :: 2$ (the diameter) : ver. sin. $(180^\circ - A)$ which is Halley's Theorem. *Phil. Trans.* No. 349. p. 466. Halley calls it "A new Theorem of good use in Trigonometry."

Cor. 1. Since $\sin. B = \frac{N}{ac}$, and $\sin. C = \frac{N}{ab}$, we have

$$\frac{\sin. B}{\sin. C} = \frac{b}{c},$$

or, if this equality be thrown into a proportion,

$$\sin. B : \sin. C :: b : c.$$

This relation, however, of the sines of the angles to the sides opposite, may be immediately deduced from (Fig. 3.); for the perpendicular (q) on a from the vertex of A, is the sine of B to the radius c, or $q = c \sin. B$ (radius $= 1$); similarly, q is the sine of C to the radius b, or, $q = b \sin. C$ (rad. 1.); $\therefore$ since $q = q$, $c \sin. B = b \sin. C$.

Cor. 2. Hence the area of a triangle may be expressed in terms of its sides, for, (see Fig. 3.) area $= \frac{a . perp.}{2} = \frac{ac}{2} . \sin. B =$

$$\sqrt{\left\{ \begin{array}{c} \left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - a\right) \\ \left(\frac{a+b+c}{2} - b\right)\left(\frac{a+b+c}{2} - c\right) \end{array} \right\}}$$

Problem 3. It is required to express the sine of the sum of two arcs, in terms of the sines and cosines of the simple arcs.

From the two preceding Problems,

$$\sin. A = \frac{N}{bc}, \qquad \sin. B = \frac{N}{ac},$$

$$\text{and, } \cos. B = \frac{a^2 + c^2 - b^2}{2ac}, \ \cos. A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$\text{Hence, } \sin. A . \cos. B + \cos. A . \sin. B = \frac{N(a^2 + c^2 - b^2 + b^2 + c^2 - a^2)}{2abc^2}$$

$$= \frac{2Nc^2}{2abc^2} = \frac{N}{ab}.$$

But $\sin. C = \frac{N}{ab}$, and $\sin. C = \sin. \{\pi - (A+B)\} = \sin. (A+B)$, for $A + B$ is supplement to $\pi - (A + B)$ as $A + B + C = \pi$.

Hence $\sin. (A+B) = \sin. A . \cos. B + \cos. A . \sin. B.$ [1]*

---

* It has been objected, and rightly, that this formula is not thus obtained

This is the fundamental form, from which almost all other Trigonometrical forms may be deduced.

Cor. 1.

$$\cos.\,(A + B) = \text{(p. 6.)}\ \sin.\left(\frac{\pi}{2} - (A + B)\right) =$$

---

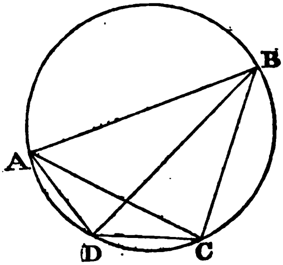
obtained by *direct* investigation, but discovered to be true by observation on the values of sin. A, cos. A, sin. B, cos. B, and by previous knowledge of those values. There is no difficulty in directly investigating it, by means of a construction (see Thomas Simpson's *Trig*. p. 54.) An able critic, however, (see *Edin. Review*, No. XXXIII. p. 128.) suggests, that this, and similar formulæ, may be naturally and simply deduced by means of a Lemma of Ptolemy in his μεγαλη συνταξις, and which in Simpson's edition of *Euclid* is the Prop. *D* of the Sixth Book. Thus, according to that proposition,

$$AB \times CD + BC \times AD = AC \times BD.$$

Let the arc $AD = DC = 2B$, and the arc $BAD = 2A$;

then,

$(AB + BC)\,AD = AC \times BD$, or, (see p. 10.)



$$\{2.\sin.(A-B) + 2\sin.(A+B)\}\ 2\sin.B = 2\sin.2B \times 2\sin.A\ (a).$$

Let $A = \frac{\pi}{2}$, then,

$$\left\{\sin.\left(\frac{\pi}{2} - B\right) + \sin.\left(\frac{\pi}{2} + B\right)\right\}\ \sin.B = \sin.2B \times \sin.\frac{\pi}{2};$$

but, (pp. 6, 13.) $\sin.\left(\frac{\pi}{2} - B,\right)$ and $\sin.\left(\frac{\pi}{2} + B\right) = \cos.B$ and $\sin.\frac{\pi}{2} = 1$;

$\therefore 2.\cos.B \times \sin.B = \sin.2B$, we have (the same as in p. 12.) and substituting this in the form (a) l. 18. we have

sin.

p. 13.) $\sin.\left(\frac{\pi}{2} + (A+B)\right)$;

$$\text{but, } \sin.\left(\frac{\pi}{2} + (A+B)\right) = \sin.\left\{\left(\frac{\pi}{2}+A\right) + B\right\}.$$

$$\sin.(A - B) + \sin.(A + B) = 2\sin. A . \cos. B \qquad (b).$$

Since in this formula A and B may be any arcs provided that $A > B$, substitute $\frac{\pi}{2} + A$ instead of A, and $\frac{\pi}{2} + B$ instead of B, then

$$\sin.(A - B) + \sin.(\pi + A + B) = 2 . \sin.\left(\frac{\pi}{2} + A\right) \cos.\left(\frac{\pi}{2} + B\right)$$

But, by Table, p. 16, and pp. 6, 13.

$$\sin.(\pi + A + B) = -\sin.(A + B);\ \sin.\left(\frac{\pi}{2} + A\right) = \cos. A,$$

$$\text{and } \cos.\left(\frac{\pi}{2} + B\right) = -\sin. B;\ \text{therefore}$$

$$\sin.(A - B) - \sin.(A + B) = -2\cos. A . \sin. B \qquad (c).$$

Hence, by adding and subtracting (b) and (c), there will result

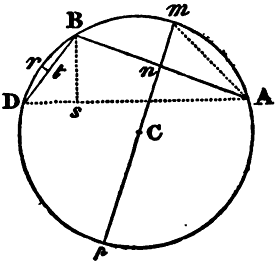
$$\sin.(A - B) = \sin. A . \cos. B - \cos. A . \sin. B,$$

$$\text{and } \sin.(A + B) = \sin. A . \cos. B + \cos. A . \sin. B,$$

and the deduction of other forms from these fundamental ones may be conducted as it is in the text.

Mr. Cresswell, in his Treatise on Spherics, has more simply demonstrated the latter of these fundamental formulæ: thus

$$\text{let } AB = 2A,\ BD = 2B,$$



and draw Bs perpendicularly to AD, then

$$AD = As + sD;$$

but

$$= \sin.\left(\frac{\pi}{2} + A\right)\cos. B + \cos.\left(\frac{\pi}{2} + A\right).\sin. B,$$

which is derived from the form [1], by substituting, instead of A, $\left(\frac{\pi}{2} + A\right)$.

But by pp. 6, 13. $\sin.\left(\frac{\pi}{2} + A\right) = \cos. A$,

$$\text{and } \cos.\left(\frac{\pi}{2} + A\right) = -\sin. A.$$

Hence, $\cos.(A + B) = \cos. A . \cos. B - \sin. A . \sin. B$....[2].

Cor. 2. By page 6,

$$\sin.(A - B) = \cos.\left(\frac{\pi}{2} - (A - B)\right) = \cos.\left\{\left(\frac{\pi}{2} - A\right) + B\right\}.$$

But, by the formula [2] that has been just established,

$$\cos.\left\{\left(\frac{\pi}{2} - A\right) + B\right\} =$$

$$\cos.\left(\frac{\pi}{2} - A\right)\cos. B - \sin.\left(\frac{\pi}{2} - A\right)\sin. B$$

$$= \sin. A . \cos. B - \cos. A . \sin. B,$$

(by p. 6.)

Hence, therefore,

$\sin.(A - B) = \sin. A \cos. B - \cos. A \sin. B$ .... [3].

---

but (see p. 10.) $AD = 2\sin.\frac{AB + BD}{2} = 2\sin.(A + B)$,

$As = AB . \cos. BAD = AB \cos. \frac{BD}{2}$ (Euclid, Book III. Prop. 20.)

$$= 2\,An . \cos. B$$

$$= 2\sin. A . \cos. B,$$

$Ds = BD . \cos. BDA = 2\sin. B \cos. A$;

$$\therefore \sin.(A + B) = \sin. A \cos. B + \cos. A . \sin. B.$$

Again, by page 6,

$$\cos.(A-B) = \sin.\left(\frac{\pi}{2} - (A-B)\right) = \sin.\left\{\left(\frac{\pi}{2} - A\right) + B\right\}.$$

But, by the formula [1],

$$\sin.\left\{\left(\frac{\pi}{2} - A\right) + B\right\} = \sin.\left(\frac{\pi}{2} - A\right)\cos.B + \cos.\left(\frac{\pi}{2} - A\right)\sin.B$$
$$= \cos.A\cos.B + \sin.A.\sin.B \text{ (by p. 6.)}$$

Hence, therefore,

*$\cos.(A-B) = \cos.A..\cos.B + \sin.A.\sin.B \ldots\ldots\ldots$ [4].

Cor. 3. Add together the forms [1] and [3,] and there results $\sin.(A+B) + \sin.(A-B) = 2.\sin.A.\cos.B \ldots\ldots$ [*a*].

Subtract [3] from [1], and

$\sin.(A+B) - \sin.(A-B) = 2.\cos.A.\sin.B \ldots\ldots$ [*b*].

Multiply [1] and [3],

and, the right-hand side of the equation is =

$$\sin.^2 A \times \cos.^2 B - \cos.^2 A \times \sin.^2 B =$$
$$\sin.^2 A(1 - \sin.^2 B) - (1 - \sin.^2 A)\sin.^2 B =$$
$$\sin.^2 A - \sin.^2 B.$$

Hence,

$\sin.(A+B) \times \sin.(A-B) = \sin.^2 A - \sin.^2 B \ldots\ldots$ [*c*].

Add [2] and [4], and

$\cos.(A-B) + \cos.(A+B) = 2\cos.A.\cos.B \ldots\ldots\ldots$ [*d*].

---

* We may from these formulæ easily derive expressions for the sine and cosine of $A \pm B \pm C$:

thus $\sin.(A+B+C) = \sin.(A+B)\cos.C + \cos.(A+B).\sin.C$
$= \sin.A\cos.B\cos.C + \cos.A\sin.B\cos.C$
$+ \cos.A\cos.B\sin.C - \sin.A\sin.B\sin.C.$

Subtract [2] from [4], and

$$\cos.(A - B) - \cos.(A + B) = 2\sin.A.\sin.B\ldots\ldots[e].$$

If we substitute in the preceding formulæ [*a*], [*b*], [*c*], &c. the quantity $(n + 1)B$ instead of A, we shall have

$$\begin{aligned}
\sin.(n+2)B + \sin.nB &= 2\sin.(n+1)B\cos.B,\\
\sin.(n+2)B - \sin.nB &= 2\cos.(n+1)B\sin.B,\\
\sin.(n+2)B \times \sin.nB &= \sin.^2(n+1)B - \sin.^2B,\\
\cos.nB + \cos.(n+2)B &= 2\cos.(n+1)B.\cos.B,\\
\cos.nB - \cos.(n+2)B &= 2\sin.(n+1)B.\sin.B.
\end{aligned}$$

COR. 4. Some of the preceding forms may be differently expressed, for

$$\left.\begin{aligned}
\text{since}\quad A &= \frac{A+B}{2} + \frac{A-B}{2} = \frac{S}{2} + \frac{D}{2}\\
\text{and}\quad B &= \frac{A+B}{2} - \frac{A-B}{2} = \frac{S}{2} - \frac{D}{2}
\end{aligned}\right\}\begin{aligned}&\text{making}\\ &S = A + B,\\ &D = A - B,\end{aligned}$$

we have from [*a*],

$$\sin.S + \sin.D = 2\sin.\left(\frac{S+D}{2}\right).\cos.\left(\frac{S-D}{2}\right),$$

and from [*b*],

$$\sin.S - \sin.D = 2\cos.\left(\frac{S+D}{2}\right).\sin.\left(\frac{S-D}{2}\right),$$

or, since S and D are any arcs subject to this condition alone, namely, that $S > D$, and since, in a series of formulæ, it is convenient to use the same characters, instead of S and D we may use A and B, and then,

$$\sin.A + \sin.B = 2\sin.\left(\frac{A+B}{2}\right).\cos.\left(\frac{A-B}{2}\right)\ldots\ldots[5],$$

$$\sin.A - \sin.B = 2\cos.\left(\frac{A+B}{2}\right).\sin.\left(\frac{A-B}{2}\right)\ldots\ldots[6].$$

By a similar process we may transform [*d*] and [*e*] into these,

$$\cos. A + \cos. B = 2\cos.\left(\frac{A+B}{2}\right).\cos.\left(\frac{A-B}{2}\right)\ldots.[7].$$

$$\cos. B - \cos. A = 2\sin.\left(\frac{A+B}{2}\right).\sin.\left(\frac{A-B}{2}\right)\ldots\ldots[8].$$

Cor. 5. Divide [5] by [6], and

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\sin.\left(\frac{A+B}{2}\right)}{\cos.\left(\frac{A+B}{2}\right)}.\frac{\cos.\left(\frac{A-B}{2}\right)}{\sin.\left(\frac{A-B}{2}\right)}.$$

$$= \tan.\left(\frac{A+B}{2}\right).\cot.\left(\frac{A-B}{2}\right),$$

$$\text{or,} = \frac{\tan.\left(\frac{A+B}{2}\right)}{\tan.\left(\frac{A-B}{2}\right)}\ldots\ldots\ldots\ldots\ldots.[f].$$

similarly, $\frac{\cos. A + \cos. B}{\cos. B - \cos. A} = \frac{1}{\tan.\left(\frac{A+B}{2}\right).\tan.\left(\frac{A-B}{2}\right)}\ldots[g]$,

and, $\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan.\left(\frac{A+B}{2}\right)\ldots\ldots\ldots\ldots\ldots\ldots[h]$,

which, in a particular case, that is, when $B = 0$, becomes

$\frac{\sin. A}{1+\cos. A} = \tan.\frac{A}{2}$, or $\frac{\sqrt{1-\cos.^2 A}}{1+\cos. A}$, or $\sqrt{\frac{1-\cos. A}{1+\cos. A}} = \tan.\frac{A}{2}$.

From this last expression, we may express, cos. A in terms of the tan. $\frac{A}{2}$, &c.; for, since

$$\tan.^2\frac{A}{2} = \frac{1-\cos. A}{1+\cos. A}, \text{ we have}$$

$$\cos. A = \frac{1-\tan.^2\frac{A}{2}}{1+\tan.^2\frac{A}{2}}.$$

⁂ **Cor.** 6. The forms for the sine and cosine of $A \pm B$ being obtained, it is easy to deduce from them, the sine or cosine of any arc, such as $\frac{m\pi}{n} \pm A$, and nearly with as much convenience as by reference to the Table given in page 16, thus, by......[1],

$$\sin.(\pi + A) = \sin.\pi.\cos.A + \cos.\pi.\sin.A,$$

$$\text{but } \sin.\pi = 0, \text{ and } \cos.\pi = -1;$$

$$\therefore \sin.(\pi + A) = -\sin.A.$$

Again, $\sin.\left(\frac{3\pi}{2} - A\right) = \sin.\frac{3\pi}{2}\cos.A - \cos.\frac{3\pi}{2}\sin.A;$

but, $\sin.\left(\frac{3\pi}{2}\right) = \sin.\left(\pi + \frac{\pi}{2}\right) = \sin.\pi.\cos.\frac{\pi}{2} + \cos.\pi.\sin.\frac{\pi}{2}$

$$= -1, \text{ since } \cos.\frac{\pi}{2} = 0, \text{ and } \cos.\pi = -1,$$

$$\cos.\frac{3\pi}{2} = \cos.\left(\pi + \frac{\pi}{2}\right) = \cos.\pi.\cos.\frac{\pi}{2} - \sin.\pi\,\sin.\frac{\pi}{2} = 0.$$

The $\sin.\left(\frac{3\pi}{2} - A\right)$, therefore, $= -\cos.A$.

Again, $\cos.\left(\frac{3\pi}{2} - 3A\right) = \cos.\frac{3\pi}{2}\cos.3A + \sin.\frac{3\pi}{2}\sin.3A$

$$= -\sin.3A, \text{ since } \cos.\frac{3\pi}{2} = 0, \text{ and } \sin.\frac{3\pi}{2} = -1.$$

Again, $\cos.\left(\frac{5\pi}{2} + A\right) = \cos.\frac{5\pi}{2}\cos.A - \sin.\frac{5\pi}{2}\sin.A,$

but $\cos.\frac{5\pi}{2} = \cos.\left(2\pi + \frac{\pi}{2}\right) = \cos.2\pi.\cos.\frac{\pi}{2} - \sin.2\pi.\sin.\frac{\pi}{2} = 0,$

$\sin.\frac{5\pi}{2} = \sin.\left(2\pi + \frac{\pi}{2}\right) = \sin.2\pi.\cos.\frac{\pi}{2} + \cos.2.\pi.\sin.\frac{\pi}{2} = 1;$

$$\therefore \cos.\left(\frac{5\pi}{2} + A\right) = -\sin.A.$$

and, in like manner, other instances may be reduced ⁂.

We will now proceed to deduce expressions for the tangents of the sums and differences of arcs: and, the first step will be the solution of the following Problem, which, indeed, is little else than a corollary from the preceding results:

PROBLEM 4. It is required to express the tangent of the sum and difference of two arcs in terms of the tangents of the simple arcs.

Since (p. 8.) the tangent of an arc is equal to the sine divided by the cosine, if the arc be $A \pm B$,

$$\tan.\,(A \pm B) = \frac{\sin.\,(A \pm B)}{\cos.\,(A \pm\,)B} = \frac{\sin.\,A\,.\cos.\,B \pm \cos.\,A\,.\sin.\,B}{\cos.\,A\,.\cos.\,B \mp \sin.\,A\,.\sin.\,B}.$$

Now, since the object is to obtain an expression involving $\tan.\,A$, $\tan.\,B$, we must divide both numerator and denominator of the above fraction by $\cos.\,A\,.\cos.\,B$, an operation which will not change its real value; beginning then with the numerator

$$\frac{\sin.\,A\,.\cos.\,B \pm \cos.\,A\,.\sin.\,B}{\cos.\,A\,.\cos.\,B} = \frac{\sin.\,A}{\cos.\,A} \pm \frac{\sin.\,B}{\cos.\,B} = \tan.\,A \pm \tan.\,B$$

$$\frac{\cos.\,A\,\cos.\,B \mp \sin.\,A\,\sin.\,B}{\cos.\,A\,.\cos.\,B} = 1 \mp \frac{\sin.\,A\,\sin.\,B}{\cos.\,A\,\cos.\,B} = 1 \mp \tan.\,A.\,\tan.\,B$$

consequently, $\tan.\,(A \pm B) = \dfrac{\tan.\,A \pm \tan.\,B}{1 \mp \tan.\,A\,.\tan.\,B}$ (rad. $= 1$) [9].

This formula may be used for determining the tangents of such arcs as $90^\circ \pm A$, $180^\circ \pm A$, &c. exactly, as in Cor. 6. p. 34, we shewed the formulæ for $\sin.\,(A \pm B)$, &c. might be used in determining $\sin.\left(\frac{m}{n}\pi \pm A\right)$: for instance,

$$\tan.\,(90^\circ + A) = \frac{\tan.\,90^\circ + \tan.\,A}{1 - \tan.\,90^\circ \tan.\,A} = \frac{\infty + \tan.\,A}{1 - \infty\,\tan.\,A} = \frac{\infty}{-\infty\,\tan.\,A}$$

$= -\dfrac{1}{\tan.\,A}$: again, $\tan.\,(180^\circ - A) = \dfrac{\tan.\,180^\circ - \tan.\,A}{1 + \tan.\,180^\circ \tan.\,A} = \dfrac{0 - \tan.\,A}{1 + 0\tan.\,A}$

$= -\tan.\,A$.

COR. 2. Since $\tan.\,45^\circ = \dfrac{\sin.\,45^\circ}{\cos.\,45^\circ} = 1$, if we make $A = 45^\circ$, there will result from the preceding expression,

$$\tan.(45^\circ \pm B) = \frac{1 \pm \tan. B}{1 \mp \tan. B}.$$

Cor. 3. Let t, t', t'', &c. be the tangents of the arcs A, B, C, &c. then by formula [9], considering $A + B$ as one arc,

$$\tan.(A + B + C) = \frac{\tan.(A + B) + \tan. C}{1 - \tan.(A+B)\tan. C} \ldots\ldots\ldots [q].$$

But, $\tan.(A + B) = \dfrac{t + t'}{1 - t t'}$;

therefore the numerator of the fraction [q] equals

$$\frac{t + t' + t'' - t t' t'')}{1 - t t'},$$

and the denominator, of the same fraction, equals

$$\frac{1 - (t t' + t t'' + t' t''}{1 - t t'}.$$

Hence,

$$\tan.(A + B + C) = \frac{t + t' + t'' - t t' t''}{1 - (t t' + t t'' + t' t'')}.$$

If $A + B + C = \pi$, (which is the case when A, B, C, are the three angles of a triangle), since $\tan. \pi = 0$,

$$t + t' + t'' - t t' t'' = 0, \text{ or}$$
$$t + t' + t'' = t t' t'',$$

which is the theorem given in the *Phil. Trans.* 1808. p. 122.

But the theorem has an origin much more remote; for, the above formula for $\tan.(A + B + C)$ and similar formulæ for the tangents of $A + B + C + D$, &c. were given as far back as the year 1722, by John Bernoulli, and are inserted in the *Leipsic Acts* for that year, p. 361, and in the second volume of his Works, at p. 526.

The formulæ for the tangents of the sums of any arcs A, B, C, &c. are symmetrical in their composition, and their law is easily defined: suppose, the symbols $S_3\,(t t')$, $S_4\,(t t' t'')$, &c. be made to represent, respectively,

$$t\,t' + t\,t'' + t'\,t'',$$

$$t\,t'\,t'' + t\,t'\,t''' + t\,t''\,t''' + t'\,t''\,t''', \text{ \&c.}$$

then, $\tan.(A + B + C) = \dfrac{S_3(t) - S_3(t\,t'\,t'')}{1 - S_3(t\,t')}$,

$$\tan.(A + B + C + D) = \frac{S_4(t) - S_4(t\,t'\,t'')}{1 - S_4(t\,t') + S_4(t\,t'\,t''\,t''')},$$

$$\tan.(A + B + C + D + E) = \frac{S_5(t) - S_5(t\,t'\,t'') + S_5(t\,t' \ldots t^{\text{iv}})}{1 - S_5(t\,t') + S_5(t\,t'\,t''\,t''')},$$

&c.

These formulæ are easily shewn to be true on the principle, that, if the formula for the tangent of n arcs be true, that for $(n + 1)$ arcs must be true also: the latter inference being made by means of the form [9], p. 35.

If, instead of a radius $= 1$, we would introduce a radius $= r$ into the preceding formulæ, we must avail ourselves of the rule laid down in p. 19. Thus the formula [1], p. 27, becomes

$$r.\sin.(A + B) = \sin. A \times \cos. B + \cos. A \times \sin. B,$$

the formula [c], p. 31, is the same, whether the radius be 1 or r.

The formula $\tan.^2 \dfrac{A}{2} = \dfrac{1 - \cos. A}{1 + \cos. A}$ to radius $= 1$

becomes $\tan.^2 \dfrac{A}{2} = \dfrac{r^3 - r^2.\cos. A}{r + \cos. A}$ when radius $= r$.

The formula [h] becomes

$$r \times \frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan.\left(\frac{A + B}{2}\right);$$

but, the formula [f], p. 33, remains the same, whether the radius be equal 1 or r.

⁂ We will now subjoin a few additional formulæ for the sines and cosines, &c. of the sums and differences of arcs, the investigation of which the Student, by pursuing a track similar to what has been already proceeded on, will easily discover,

$$\text{sin.}\,(45^{0} \pm A) = \frac{\text{cos.}\,A \pm \text{sin.}\,A}{\sqrt{2}},$$

$$\text{cos.}\,(45^{0} \mp A) = \frac{\text{cos.}\,A \pm \text{sin.}\,A}{\sqrt{2}},$$

$$\text{sin.}\,(60^{0} + A) - \text{sin.}\,(60^{0} - A) = \text{sin.}\,A,$$

$$\text{cos.}\,(60^{0} + A) + \text{cos.}\,(60^{0} - A) = \text{cos.}\,A,$$

$$\text{tan.}\,(45^{0} + A) = \frac{1 + \text{tan.}\,A}{1 - \text{tan.}\,A},$$

$$\text{tan.}\,(45^{0} - A) = \frac{1 - \text{tan.}\,A}{1 + \text{tan.}\,A},$$

$$\text{tan.}^{2}\,A - \text{tan.}^{2}\,B = \frac{\text{sin.}\,(A - B)\,\text{sin.}\,(A + B)}{\text{cos.}^{2}\,A \times \text{cos.}^{2}\,B},$$

$$\text{cot.}^{2}\,B - \text{cot.}^{2}\,A = \frac{\text{sin.}\,(A - B)\,\text{sin.}\,(A + B)}{\text{sin.}^{2}\,A \times \text{sin.}^{2}\,B}.$$

M. Cagnoli, in his *Trigonometry*, has collected into a Table, under one view, and for the purpose of reference, formulæ similar to the preceding. He has also in another Table (which is subjoined) exhibited the various values for the sine, cosine, and tangent of the angle A.

Table.

	Values of sin. A.		Values of the cos. A.
1.	$\cos. A \tan. A.$	1.	$\frac{\sin. A}{\tan. A}.$
2.	$\frac{\cos. A}{\cot. A}.$	2.	$\sin. A \cot. A.$
3.	$\sqrt{(1-\cos.^2 A)}.$	3.	$\sqrt{(1-\sin.^2 A)}.$
4.	$\frac{1}{\sqrt{(1+\cot.^2 A)}}.$	4.	$\frac{1}{\sqrt{(1+\tan.^2 A)}}.$
5.	$\frac{\tan. A}{\sqrt{(1+\tan.^2 A)}}.$	5.	$\frac{\cot. A}{\sqrt{(1+\cot.^2 A)}}.$
6.	$2 \sin. \frac{A}{2} . \cos. \frac{A}{2}.$	6.	$\cos.^2 \frac{A}{2} - \sin.^2 \frac{A}{2}.$
7.	$\sqrt{\left(\frac{1-\cos. 2A}{2}\right)}.$	7.	$1-2 \sin.^2 \frac{A}{2}.$
8.	$\frac{2 \tan. \frac{A}{2}}{1+\tan.^2 \frac{A}{2}}.$	8.	$2 . \cos.^2 \frac{A}{2} - 1.$
9.	$\frac{2}{\cot. \frac{A}{2} + \tan. \frac{A}{2}}.$	9.	$\sqrt{\left(\frac{1+\cos. 2A}{2}\right)}.$
10.	$\frac{\sin. (30^\circ + A) - \sin. (30^\circ - A)}{\sqrt{3}}.$	10.	$\frac{1-\tan.^2 \frac{A}{2}}{1+\tan.^2 \frac{A}{2}}.$
11.	$2 \sin.^2 \left(45^\circ + \frac{A}{2}\right) - 1.$	11.	$\frac{\cot. \frac{A}{2} - \tan. \frac{A}{2}}{\cot. \frac{A}{2} + \tan. \frac{A}{2}}.$
12.	$1 - 2 \sin.^2 \left(45^\circ - \frac{A}{2}\right).$	12.	$\frac{1}{1+\tan. A \tan. \frac{A}{2}}.$
13.	$\frac{1-\tan.^2 \left(45^\circ - \frac{A}{2}\right)}{1+\tan.^2 \left(45^\circ - \frac{A}{2}\right)}.$	13.	$\frac{2}{\tan. \left(45^\circ + \frac{A}{2}\right) + \cot. \left(45^\circ + \frac{A}{2}\right)}$
14.	$\frac{\tan. \left(45^\circ + \frac{A}{2}\right) - \tan. \left(45^\circ - \frac{A}{2}\right)}{\tan. \left(45^\circ + \frac{A}{2}\right) + \tan. \left(45^\circ - \frac{A}{2}\right)}$	14.	$2 \cos. \left(45^\circ + \frac{A}{2}\right) \cos. \left(45^\circ - \frac{A}{2}\right)$
15.	$\sin. (60^\circ + A) - \cos. (60^\circ - A).$	15.	$\cos. (60^\circ + A) - \cos. (60^\circ - A).$

Table.

Values of tan. A.

1. $\dfrac{\sin. A}{\cos. A}$.

2. $\dfrac{1}{\cot. A}$.

3. $\sqrt{\left(\dfrac{1}{\cos.^2 A} - 1\right)}$.

4. $\dfrac{\sin. A}{\sqrt{(1 - \sin.^2 A)}}$.

5. $\dfrac{\sqrt{(1 - \cos.^2 A)}}{\cos. A}$.

6. $\dfrac{2 \tan. \dfrac{A}{2}}{1 - \tan.^2 \dfrac{A}{2}}$.

7. $\dfrac{2 \cot. \dfrac{A}{2}}{\cot.^2 \dfrac{A}{2} - 1}$.

8. $\dfrac{2}{\cot. \dfrac{A}{2} - \tan. \dfrac{A}{2}}$.

9. $\cot. A - 2 \cot. 2 A$.

10. $\dfrac{1 - \cos. 2 A}{\sin. 2 A}$.

11. $\dfrac{\sin. 2 A}{1 + \cos. 2 A}$.

12. $\sqrt{\left(\dfrac{1 - \cos. 2 A}{1 + \cos. 2 A}\right)}$.

13. $\dfrac{\tan. \left(45^\circ + \dfrac{A}{2}\right) - \tan. \left(45^\circ - \dfrac{A}{2}\right)}{2}$ ⁂.

The investigation of some of these expressions has been already given; and, by pursuing its plan, the Student will, without difficulty, be able to accomplish that of the others.

But the Student, whose object is utility, will feel averse from their investigation, should he suspect them to be mere Trigonometrical curiosities. Such however is not their character; on the contrary, they, in many instances, materially expedite calculation, and furnish to the general language of analysis convenient forms and modes of expression. It is, in accomplishing this latter purpose, that Trigonometrical formulæ are chiefly useful: they serve to conduct investigation where the object has no concern whatever with the properties of triangles.

Yet, the investigation of the properties of triangles was the object for which Trigonometry was originally invented; and, if the Student purposes to limit his enquiries to that object alone, he need not, in quest of the requisite formulæ, advance farther than the present Chapter. He may immediately pass on to the fifth Chapter and apply what he has already learned. If, however, his views should extend farther, and he should wish to be possessed of Trigonometry and its formulæ as instruments of language, he must pursue his researches, become conversant with expressions merely analytical, and, for a time, defer their application.

In order that this latter plan may be adopted, we will, in the next Chapter, continue the deduction of Trigonometrical formulæ.

CHAP. III

On the Sines, Cosines, &c. of multiple Arcs.—Powers of the Sine and Cosine of the simple Arc.—Series of the Cosines of Arcs in Arithmetical Progression.—Vieta's, Waring's, and Cotes's Properties of Curves.—De Moivre's Expression for the Sine and Cosine of a multiple Arc by means of imaginary Symbols.

PROBLEM 5. IT is required to express the sine and cosine of twice an arc, in terms of the sine and cosine of the simple arc.

By form [1], p. 27,

$$\sin.(A+B) = \sin.A.\cos.B + \cos.A.\sin.B.$$

Let $B = A$;

$\therefore \sin.(2A) = \sin.A.\cos.A + \cos.A.\sin.A = 2\sin.A\cos.A$;

or, by the Rule of p. 19, to rad^s. r, $r\sin.2A = 2\sin.A.\cos.A$.*

Again, by the form [2] of p. 27,

$$\cos.(A+B) = \cos.A.\cos.B - \sin.A.\sin.B.$$

Let $A = B$; $\therefore \cos.2A = \cos.^2 A - \sin.^2 A$;

$$\text{or, } = \cos.^2 A - (1 - \cos.^2 A) = 2.\cos.^2 A - 1;$$

$$\text{or, } = 1 - \sin.^2 A - \sin.^2 A = 1 - 2\sin.^2 A.$$

If we employ a radius $= r$, then, by the Rule of p. 19,

$r.\cos.2A = 2\cos.^2 A - r^2$, or $= r^2 - 2.\sin.^2 A$, (see p. 11.)

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* This result has been (p. 12.) already obtained, but it is here repeated, as being the first of a series of formulæ deduced on the same principle.

Cor. 1. By transposition,

$$r^2 - r.\cos.2A, \text{ or } r(r - \cos.2A) = 2\sin.^2 A \text{ (see p. 11.)}.$$

Now, $r - \cos.2A = \text{ver. sin. }2A$, consequently,

$$r.(\text{ver. sin. }2A) = 2\sin.^2 A \text{ (see p. 11.)}$$

which equality, thrown into a proportion, is

$$\text{ver. sin. }2A : \sin.A :: \sin.A : \frac{r}{2};$$

or, expressed in general terms, announces, that the sine of an angle is a mean proportional between the versed sine of twice the same angle and the semi-radius.

$$\text{Again, } r^2 + r.\cos.2A = 2\cos.^2 A,$$
$$\text{or } r(r + \cos.2A) = 2\cos.^2 A,$$
$$\text{or } r(\text{ver. sin. supp}^t.\text{ of }2A) = 2\cos.^2 A;$$

or, (if we call the ver. sin. of the supplement of an arc, the suversed sine) $r.\text{suversin.}(2A) = 2\cos.^2 A$, which equality, like the preceding, may be expressed under the form of a proportion, or in general terms.

Cor. 2. Hence the sine of $30^\circ = \frac{1}{2}$, radius $= 1$,

For, $\sin.60^\circ = \sin.(2.30^\circ) = 2\sin.30^\circ.\cos.30^\circ$, [Prob. 5.],
but, $\cos.30^\circ = \sin(90^\circ - 30^\circ) = \sin.60^\circ$;
$\therefore \sin.60^\circ = 2\sin.30^\circ.\sin.60^\circ$;

or $\sin.30^\circ = \frac{1}{2}$, and consequently $\sin.60^\circ$, or $\cos.30^\circ = \frac{\sqrt{3}}{2}$.

Cor. 3. Since, radius being 1,

$$\sin.^2 A + \cos.^2 A = 1,$$

and, (p. 12.) $2\sin.A.\cos.A = \sin.2A$,

we have, by the solution of a quadratic equation such as

$$x^2 + y^2 = a,$$
$$xy = b,$$
$$\sin.A = \tfrac{1}{2}\sqrt{(1 + \sin.2A)} \pm \tfrac{1}{2}\sqrt{(1 - \sin.2A)},$$
$$\cos.A = \tfrac{1}{2}\sqrt{(1 + \sin.2A)} \mp \tfrac{1}{2}\sqrt{(1 - \sin.2A)},$$

in which expressions the upper or lower sign that affects the second term is to be used, accordingly as sin. A is greater or less than cos. A, that is, in arcs not exceeding a quadrant, accordingly as A is greater or less than half a quadrant.

These two formulæ are useful either to compute arithmetically the sines, cosines, &c. of arcs, or to examine their accuracy when computed by other formulæ; and, performing the latter office, they are called *Formulæ of Verification*. It is easy to perceive their use in computing sines, cosines, &c.; since, if we take sin. 2 A, a known quantity, for instance, the sin. 30° which equals $\frac{1}{2}$, we may, by successive substitutions, regularly deduce the sines of 15°, 7° 30′, &c. Thus,

$$\text{sin. } 15^\circ = \tfrac{1}{2}\sqrt{(1+\tfrac{1}{2})} - \tfrac{1}{2}\sqrt{(1-\tfrac{1}{2})} = .258819,$$

$$\text{sin. } 7^\circ\ 30' = \tfrac{1}{2}\sqrt{(1.258819)} - \tfrac{1}{2}\sqrt{(.741181)} = .1305262,$$

$$\text{sin. } 3^\circ\ 45' =, \ \&\text{c.}$$

Problem 6. It is required to express the sine and cosine of 3 times an arc, 4 times an arc, &c. in terms of the sine and cosine of the simple arc.

If we substitute in the formula for cos. $(A+B)$ (p. 29.) 2 A instead of B, we have

$$\text{cos. } (A + 2A) = \text{cos. } 2A \,.\, \text{cos. } A - \text{sin. } 2A \,.\, \text{sin. } A;$$

but, by the preceding Problem,

$$\text{sin. } 2A = 2 \text{ sin. } A \text{ cos. } A,$$

$$\text{and cos. } 2A = 2 \text{ cos.}^2 A - 1;$$

$$\begin{aligned} \therefore \text{ cos. } 3A &= (2 \,.\, \text{cos.}^2 A - 1) \text{ cos. } A - 2 \text{ cos. } A \,.\, \text{sin.}^2 A \\ &= 2 \,.\, \text{cos.}^3 A - \text{cos. } A - 2 \text{ cos. } A\,(1 - \text{cos.}^2 A) \\ &= 4 \text{ cos.}^3 A - 3 \text{ cos. } A, \text{ when the radius} = 1, \\ &= \frac{4 \text{ cos.}^3 A}{r^2} - 3 \,.\, \text{cos. } A, \text{ when the radius} = r. \end{aligned}$$

This form, if we substitute therein, instead of the arc 3 A, $\frac{3\pi}{2} - 3A$ gives us

$$\cos.\left(\frac{3\pi}{2}-3A\right)=4.\left(\cos.\frac{\pi}{2}-A\right)^3-3.\cos.\left(\frac{\pi}{2}-A\right);$$

but, by Cor. 6, Prob. 3, p. 34.

$$\cos.\left(\frac{3\pi}{2}-3A\right)=-\sin.3A,\ \cos.\left(\frac{\pi}{2}-A\right)=\sin.A;$$

consequently,

$$\sin.3A=3\sin.A-4.\sin.^3A,\text{ when the radius }=1,$$

$$=3r^2.\sin.A-\frac{4.\sin.^3A}{r^2}\text{ when the radius }=r.$$

By a similar method may the $\cos.(4A)=\cos.(3A+A)$ or $=\cos.(2A+2A)$, and the $\cos.(5A)=\cos.(4A+A)$ or $=\cos.(3A+2A)$, &c. be deduced. But, the successive formation of the cosines and sines of multiple arcs may, most easily, be effected after the following manner:

By the form [d], page 31,

$$\cos.(A+B)+\cos.(A-B)=2\cos.A.\cos.B\ldots\ldots A>B,$$

$$\text{or }\cos.(B+A)+\cos.(B-A)=2\cos.B.\cos.A\ldots\ldots B>A.$$

Let $B=nA$, then transposing

$$\cos.(n+1)A=2.\cos.nA.\cos.A-\cos.(n-1)A,$$

and hence from $\cos.(n-1)A$, and $\cos.nA$, may be assigned $\cos.(n+1)A$: for instance,

$$\text{if } n=1,\ \cos.(n-1)A=\cos.0=1;$$

$$\therefore\ \cos.2A=2\cos.^2A-1\ldots\ldots\ldots\ldots\ldots[c^{II}].$$

If $n=2$,

$$\cos.3A=2.\cos.2A.\cos.A-\cos.A=4.\cos.^3A-3\cos.A\ [c^{III}].$$

If $n=3$,

$$\cos.4A=2\cos.3A.\cos.A-\cos.2A$$
$$=2(4\cos.^3A-3\cos.A)\cos.A-(2\cos.^2A-1)$$
$$=8\cos.^4A-8\cos.^2A+1\ldots\ldots\ldots\ldots[c^{IV}],$$

And, by a like process, if $n=4$,

$$\cos.5A=$$

$$16 \cos.^5 A - 16 \cos.^3 A + 2 \cos. A - (4 \cos.^3 A - 3 \cos. A)$$
$$= 16 \cos.^5 A - 20 \cos.^3 A + 5 \cos. A \ldots\ldots\ldots\ldots [c^V],$$
$$\text{or, } 2 \cos. 5 A = (2 \cos. A)^5 - 5 (2 \cos. A)^3 + \cos. A.$$

Similarly, if $[n = 5]$,

$$\cos. 6 A = 32 \cos.^6 A - 40 \cos.^4 A + 10 \cos.^2 A$$
$$- (8 \cos.^4 A - 8 \cos.^2 A + 1)$$
$$= 32 \cos.^6 A - 48 \cos.^4 A + 18 \cos.^2 A - 1 \ldots\ldots [c^{VI}],$$

$$\text{or, } 2 . \cos. 6A = 2^6 . \cos.^6 A - 6 . 2^4 \cos.^4 A + \frac{6 . 3}{2} 2^2 \cos.^2 A - 2,$$

and the general form is

$$2 \cos. mA = (2 \cos. A)^m - m(2 \cos. A)^{m-2} + \frac{m.(m-3)}{2} . (2 \cos. A)^{m-4} \qquad [c^M]$$
$$- \frac{m.(m-4)(m-5)}{2 . 3} (2 \cos. A)^{m-6} + \&c.$$

The formula for the sines of multiple arcs may be deduced from those of the cosines, and, on the same principle as that which has been already used in deducing sin. 3 A; by substituting, in the form for cos. 5 A, $\frac{5\pi}{2} - 5A$ instead of 5 A, we have

$$\cos. \left(\frac{5\pi}{2} - 5A\right) =$$
$$16 \left(\cos. \frac{\pi}{2} - A\right)^5 - 20 \left(\cos. \frac{\pi}{2} - A\right)^3 + 5 . \cos. \left(\frac{\pi}{2} - A\right),$$

$$\text{but by Cor. 6. Prob. 3, } \cos. \left(\frac{5\pi}{2} - 5A\right) = \sin. 5 A,$$
$$\cos. \left(\frac{\pi}{2} - A\right) = \sin. A;$$

consequently, $\sin. 5 A = 16 \sin.^5 A - 20 \sin.^3 A + 5 \sin. A$.

Or, the sines of multiple arcs may be successively deduced as the cosines have, on the same principle, and by like formulæ; thus, by the form [a], p. 31.

$$\sin.(B + A) + \sin.(B - A) = 2\sin. B . \cos. A.$$

Let $B = nA$, then, transposing

$$\sin.(n + 1)A = 2\sin. nA . \cos. A - \sin.(n - 1)A,$$

hence, if $n = 2$, $\sin. 3A = 2 . \sin. 2A . \cos. A - \sin. A,$

$$(\sin. 2A = 2\sin. A . \cos. A) = 4\sin. A . \cos.^2 A - \sin. A \, . \, . \, [s^{II}]$$

$$= 3\sin. A - 4\sin.^3 A \, . \, . \, . \, . \, . \, . \, [s^{III}].$$

Similarly, if $n = 3$,

$$\begin{aligned}
\sin. 4A &= 2\cos. A\,(3\sin. A - 4\sin.^3 A) - \sin. 2A,\\
&= 6\sin. A . \cos. A - 8\cos. A . \sin.^3 A - 2\sin. A . \cos. A\\
&= (4\sin. A - 8\sin.^3 A)\cos. A,\\
\text{or, } &= (8\cos.^3 A - 4\cos. A)\sin. A \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, [s^{IV}].
\end{aligned}$$

Similarly, if $n = 4$, $\sin. 5A =$

$$\begin{aligned}
&(4\sin. A - 8\sin.^3 A)\,2\cos.^2 A - (3\sin. A - 4\sin.^3 A)\\
= {}&8\sin. A - 16\sin.^3 A - 8\sin.^3 A + 16\sin.^5 A - (3\sin. A - 4\sin.^3 A)\\
= {}&5\sin. A - 20\sin.^3 A + 16\sin.^5 A \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, . \, [s^{V}],
\end{aligned}$$

$$\text{or, } 2\sin. 5A = 5 . 2\sin. A - 5 . (2\sin. A)^3 + (2\sin. A)^5.$$

The general expression for $\sin. mA$ [m odd] is

$$\sin. mA =$$

$$m . \sin. A - \frac{m . (m^2 - 1)}{2 . 3}\sin.^3 A + \frac{m\,(m^2 - 1)(m^2 - 9)}{2 . 3 . 4 . 5}\sin.^5 A - \&c.$$

and [m even] is

$$\text{*} \sin. mA = \cos. A\left\{m . \sin. A - \frac{m . (m^2 - 4)}{2 . 3}\sin.^3 A + \frac{m . (m^2 - 4)(m^2 - 16)}{2 . 3 . 4 . 5}\sin.^5 A - \&c.\right\}.$$

The sine and cosine of the multiple arc (mA) have been expressed in powers of the sine and cosine of the simple arc (A): but, if we express the cosine of the arc (A) by a particular binomial, the cosines of the multiple arcs admit of an expression rather remarkable; or, in other words, they may be said to possess a curious property: thus, let

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* For the general demonstration of these forms, see the Appendix.

$$2 . \cos. A = x + \frac{1}{x};$$

then, $2 \cos. 2A = 2(2 \cos.^2 A - 1) = 2 \left\{ \frac{1}{2} \left(x + \frac{1}{x} \right)^2 - 1 \right\}$

$$= x^2 + \frac{1}{x^2} .$$

By the form of p. 45.

$$2 . (\cos. 3 A) = 4 \cos. 2 A . \cos. A - 2 \cos. A$$

$$= \left(x^2 + \frac{1}{x^2} \right) \left(x + \frac{1}{x} \right) - \left(x + \frac{1}{x} \right)$$

$$= x^5 + \frac{1}{x^3} .$$

Generally, if $2 . \cos. (n-1) A = x^{n-1} + \frac{1}{x^{n-1}}$,

and, $2 . \cos. n A = x^n + \frac{1}{x^n}$, then, since (p. 32,)

$$\cos. (n+1) A = 2 . \cos. n A . \cos. A - \cos. (n-1) A,$$

$$2 \cos. (n+1) A = \left(x^n + \frac{1}{x^n} \right) \left(x + \frac{1}{x} \right) - \left(x^{n-1} + \frac{1}{x^{n-1}} \right)$$

$$* = x^{n+1} + \frac{1}{x^{n+1}} .$$

Hence, if the form were true for two successive inferior numbers, $n-1$, and n, it would be true for $n+1$; but, it has been proved to be true in those cases, when $n-1 = 2$, and $n = 3$: consequently, it is true for $n+1=4$, and so on successively for all whole superior numbers.

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* This mode of denoting the cosines of multiple arcs, which leads to several curious results, occurs first in De Moivre's *Miscellanea Analytica*, pages 8 and 16.

The above expression holds good also for $\cos.\frac{n+1}{2}A$, when n is an even number. For, since

$$\cos.^2\frac{A}{2}=\frac{\cos.A+1}{2},$$

$$4.\cos.^2\frac{A}{2}=2.\cos.A+2=x+2+\frac{1}{x};$$

$$\therefore\ 2.\cos.\frac{A}{2}=\sqrt{x}+\frac{1}{\sqrt{x}}.$$

Now,

$$\cos.\frac{n+1}{2}A=2.\cos.\frac{n}{2}A\times\cos.\frac{A}{2}-\cos.\frac{n-1}{2}A;$$

$$\therefore\ 2.\cos.\frac{n+1}{2}A=\left(x^{\frac{n}{2}}+\frac{1}{x^{\frac{n}{2}}}\right)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)-\left(x^{\frac{n-1}{2}}+\frac{1}{x^{\frac{n-1}{2}}}\right)$$

$$=x^{\frac{n+1}{2}}+\frac{1}{x^{\frac{n+1}{2}}};$$

therefore, as before, if the expression be true for $\cos.\frac{n-1}{2}A$, it is (since it certainly is so for $\cos.\frac{n}{2}A$, n being an even number) true, for $\cos.\frac{n+1}{2}.A$. But, (l. 5,) it is, when $n=2$, true for $\cos.\frac{A}{2}$; therefore, it is true for $\cos.\frac{3A}{2}$; therefore for $\cos.\frac{5A}{2}$, and, by virtue of these successive inferences, generally true.

The above mode of expressing the cosines of multiple arcs is useful on several occasions: for instance, in finding the sum of a series, such as

$$\cos. A + \cos. 2A + \cos. 3A + \&c. + \cos. nA,$$

for, retaining the former notation, the sum is equal the sum of the two following series,

$$\frac{1}{2}(x + x^2 + x^3 + \&c. + x^n),$$

$$\frac{1}{2}\left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \&c. + \frac{1}{x^n}\right);$$

$$\therefore = \frac{1}{2}\left(\frac{x^{n+1} - x}{x-1} + \frac{x^n - 1)}{x^n(x-1)}\right),$$

$$= \frac{1}{2}\left(\frac{(x^n - 1)(x^{n+1} + 1)}{x^n(x-1)}\right) = \frac{1}{2}\left\{\frac{\left(x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}\right)\left(x^{\frac{n+1}{2}} + \frac{1}{x^{\frac{n+1}{2}}}\right)}{x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}}\right\}$$

Now, $\dfrac{x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}}{\sqrt{x} - \frac{1}{\sqrt{x}}} = \sqrt{\left(\dfrac{x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}}{\sqrt{x} - \frac{1}{\sqrt{x}}}\right)^2} =$

$$\sqrt{\left(\frac{\left(x^{\frac{n}{2}} + \frac{1}{x^{\frac{n}{2}}}\right)^2 - 4}{\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 4}\right)} = \sqrt{\left\{\frac{\left(2.\cos. \frac{nA}{2}\right)^2 - 4}{\left(2.\cos. \frac{A}{2}\right)^2 - 4}\right\}}$$

$= \sqrt{\left(\dfrac{1 - \cos.^2 \frac{nA}{2}}{1 - \cos.^2 \frac{A}{2}}\right)} = \dfrac{\sin. \frac{nA}{2}}{\sin. \frac{A}{2}}$: consequently the sum of the series is equal $\dfrac{\sin. \frac{nA}{2}}{\sin. \frac{A}{2}} \times \cos. \left(\dfrac{n+1}{2}\right) A.$

Since the sin. $nA = \dfrac{1}{2\sqrt{-1}}\left(x^n - \dfrac{1}{x^n}\right)$, we cannot, using the above notation, directly investigate the sum of a series such as

$$\sin. A + \sin. 2A + \sin. 3A + \&c.$$

except by introducing imaginary symbols. The investigation, however, is not difficult, and may conveniently be effected, (although by something like an artifice of calculation) in the following manner:

In the form [8], p. 33, if we substitute, instead of B and A, successively the arcs $\frac{A}{2}$ and $\frac{3A}{2}$, $\frac{3A}{2}$ and $\frac{5A}{2}$, &c. there will result

$$\cos. \frac{A}{2} - \cos. \frac{3A}{2} = 2 . \sin. A . \sin. \frac{A}{2},$$

$$\cos. \frac{3A}{2} - \cos. \frac{5A}{2} = 2 \sin. 2A . \sin. \frac{A}{2},$$

$$\cos. \frac{5A}{2} - \cos. \frac{7A}{2} = 2 \sin. 3A . \sin. \frac{A}{2},$$

$$\&c. \qquad\qquad \&c.$$

Hence, by addition,

$$\cos. \frac{A}{2} - \cos. \frac{2n+1}{2} A = 2 \sin. \frac{A}{2} (\sin. A + \sin. 2A + \sin. 3A + \&c. + \sin. nA);$$

consequently, $\sin. A + \sin. 2A + \sin. 3A + \&c. =$

$$\frac{\cos. \frac{A}{2} - \cos. \frac{2n+1}{2} A}{2 \sin. \frac{A}{2}} = \frac{\sin. \frac{n}{2} . A \times \sin. \frac{n+1}{2} A}{\sin. \frac{A}{2}}.$$

By a similar artifice may the sum of the series

$$\frac{1}{2} \tan. \frac{A}{2} + \frac{1}{4} \tan. \frac{A}{4} + \frac{1}{8} \tan. \frac{A}{8} + \&c.$$

be found.

For, if in the expression for tan. $(A + B)$ we make $A = B$,

$$\text{tan. } 2\,A = \frac{2 \text{ tan. } A}{1 - \text{tan.}^2 A}\text{, and therefore (p. 10,)}$$

$$\text{cot. } 2\,A = \frac{1 - \text{tan.}^2 A}{2 \text{ tan. } A} = \frac{1}{2 \text{ tan. } A} - \frac{\text{tan. } A}{2} =$$

$\frac{1}{2}$ cot. $A - \frac{1}{2}$ tan. A (which agrees with the 9th expression for tan. A in the Table, p. 40.).

Hence, transposing, and substituting, instead of $2\,A$, A, successively the arcs, A and $\frac{A}{2}$, $\frac{A}{2}$ and $\frac{A}{4}$, &c. there results (continually dividing each successive formula by 2),

$$\frac{1}{2} \text{ cot. } \frac{A}{2} - \text{cot. } A = \frac{1}{2} \text{ tan. } \frac{A}{2}\text{,}$$

$$\frac{1}{4} \text{ cot. } \frac{A}{4} - \frac{1}{2} \text{ cot. } \frac{A}{2} = \frac{1}{4} \text{ tan. } \frac{A}{4}\text{,}$$

$$\frac{1}{8} \text{ cot. } \frac{A}{8} - \frac{1}{4} \text{ cot. } \frac{A}{4} = \frac{1}{8} \text{ tan. } \frac{A}{8}\text{, \&c.}$$

Hence, by addition

$$\frac{1}{2^n} \text{ cot. } \frac{A}{2^n} - \text{cot. } A = \frac{1}{2} \text{ tan. } \frac{A}{2} + \frac{1}{4} \text{ tan. } \frac{A}{2} + \frac{1}{8} \text{ tan. } \frac{A}{8} + \text{\&c.}$$

By a similar process,

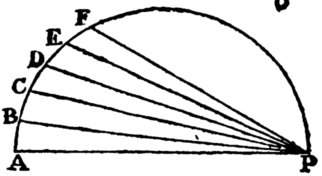
$$\text{co-sec. } A + \text{co-sec. } 2\,A + \text{co-sec. } 4A + \text{\&c.} = \text{cot. } \frac{A}{2} - \text{cot. } 2^{n-1} A.$$

See Cagnoli's *Trigonometry*, second edit. p. 122.

We will now return to the expression $2 \cos. nA = x^n + \frac{1}{x^n}$, and shew its use in demonstrating a property of curves given by Waring, in his *Proprietates Curvarum*, p. 110. The property is this:

If in a circle $ABCD$, &c. the radius of which is 1, equal arcs

AB, BC, CD, &c. be taken, and if PB be taken $= p$, p being



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the coefficient of the second term of the quadratic equation $Z^2 - pZ + 1 = 0$, of which the roots are, α, β;

$$\text{then, } PB = \alpha + \beta, \text{ or } = \alpha + \frac{1}{\alpha} \text{ since } \alpha\beta = 1,$$

$$PC = \alpha^2 + \beta^2, \text{ or } = \alpha^2 + \frac{1}{\alpha^2},$$

$$PD = \alpha^3 + \beta^3, \text{ or } = \alpha^3 + \frac{1}{\alpha^3},$$

$$\&c. = \&c. \text{ or } = \&c.$$

Now, $PB = \text{chord}\,(\pi - AB) = 2\sin.\left(\frac{\pi}{2} - \frac{AB}{2}\right) = 2\,.\cos.\frac{AB}{2}$

similarly, $PC = \text{chord}\,(\pi - AC) = 2\sin.\left(\frac{\pi}{2} - \frac{AC}{2}\right) = 2\,.\cos.\frac{AC}{2}$,

$$PD = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 2\cos.\frac{AD}{2},$$

$$PE = \&c.$$

and it has been already proved, that if, (putting $\frac{AB}{2}$ for A),

$$2\,.\cos.\frac{AB}{2} = x + \frac{1}{x};$$

$$\text{then } 2\,.\cos.2\left(\frac{AB}{2}\right), \text{ or } 2\cos.\left(\frac{AC}{2}\right) = x^2 + \frac{1}{x^2};$$

$$2\,.\cos.3\left(\frac{AB}{2}\right), \text{ or } 2\cdot\cos.\frac{AD}{2} = x^3 + \frac{1}{x^3}, \text{ \&c.}$$

hence, PB being $x + \frac{1}{x}$, or $\alpha + \frac{1}{\alpha}$, or, $\alpha + \beta$,

$$\begin{aligned} PC &= \alpha^2 + \beta^2, \\ PD &= \alpha^3 + \beta^3, \\ \&c. &= \&c. \end{aligned}$$

Vieta, p. 295, *Opera Mathematica*, Leyden, 1646, expresses the values of these chords, not by the sums of the powers of the roots, but by expressions equivalent to such sums: thus, he puts for PB, $1\,N$, or N; $(N)^2$ he represents by $1\,Q$, $(N)^3$ by $1\,C$, $(N)^5$ by $1\,QC$, &c. then

$$\left.\begin{array}{l} 1\,Q-2=PC \\ 1\,C-3N=PD \\ 1\,QQ-4Q+2=PE \\ 1\,QC-5C+5N=PF \\ \quad\text{[*] \&c.} \end{array}\right\} \text{or in modern notation,} \left\{\begin{array}{l} N^2-2=PC \\ N^3-3N=PD \\ N^4-4N^2+2=PE \\ N^5-5N^3+5N=PF, \\ \quad\text{\&c.} \end{array}\right.$$

but, $N^2 - 2$, $N^3 - 3N$, &c. express the sums, of the squares, of the cubes, &c. of the roots of an equation $x^2 - Nx + 1$; for, the formula for the sum of the m^{th} powers is

$$N^m - mN^{m-2} + m.\frac{m-3}{2}N^{m-4} - m.\frac{m-4}{2}.\frac{m-5}{3}N^{m-6} + \&c.$$

Vieta, therefore, is not to be entirely excluded from the honour due to the invention of the preceding theorem.

Vieta calculated by chords; and, his formulæ, which we have just given, are, in fact, the same as the expressions for cos. $2A$, cos. $3A$, &c. given in pages 42 and 44.

Vieta also has, p. 297, given another form, exhibiting the relations between the chords of AB, AC, AD, &c. He puts the chord of $AB = 1$ and the relation of the chord of AC to the chord of AB, N, consequently, $N = \frac{\text{chord } AC}{\text{chord } AB} =$

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[*] See Simpson's *Essays*, p. 106.

$$\frac{2\,.\,\sin.\frac{AC}{2}}{2\,.\,\sin.\frac{AB}{2}}=\frac{\sin.AB}{\sin.\frac{AB}{2}}=\frac{2\sin.\frac{AB}{2}\,.\,\cos.\frac{AB}{2}}{\sin.\frac{AB}{2}}=2\cos.\frac{AB}{2}.$$

He then forms this Table:

$$\left.\begin{array}{l} 1\,Q-1=\text{chord }AD\\ 1\,C-2N=\text{chord }AE\\ 1QQ-3Q+1=\text{chord }AF\\ \quad\&\text{c.}\end{array}\right\}\begin{array}{c}\text{or in}\\ \text{modern}\\ \text{notation,}\end{array}\left\{\begin{array}{l} N^2-1=\text{chord }AD\\ N^3-2N=\text{chord }AE\\ N^4-3N^2+1=\text{chord }AF\\ \quad\&\text{c.}\end{array}\right.$$

Now since the chord $AD=2\sin.\,3\left(\frac{AB}{2}\right)$, and the chord $AE=2\sin.\,4\left(\frac{AB}{2}\right)$, &c. and since $2\sin.\frac{AB}{2}$ is put $=1$, the preceding formulæ become, if we put $\frac{AB}{2}=A$,

$$(2\cos.A)^2-1=2\sin.3A,$$
$$(2\cos.A)^3-2\times2\,.\cos.A=2\,.\sin.4A,$$
&c.

which are the same, in fact, as $[s^{II}]$ $[s^{IV}]$ given in page 47.

We may also employ the above mode of expressing the cosines of multiple arcs, in deducing de Moivre's formula, which is

$$^{*}(\cos.A+\sqrt{-1}.\sin.A)^m=\cos.mA+\sqrt{-1}.\sin.mA,$$

for since

$$\cos.A=\frac{1}{2}\left(x+\frac{1}{x}\right),\ \sin.A=\frac{1}{2}\sqrt{\left(-x^2+2-\frac{1}{x^2}\right)}$$

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* Lagrange, p. 116 *Calcul des Fonctions*, says, that this form is as remarkable for its simplicity and elegance, as it is for its generality and utility: and M. Laplace, in the *Leçons des Ecoles Normales*, considers the invention of this formula to be of equal importance with that of the Binomial Theorem.

$$= \frac{\sqrt{-1}}{2}\left(x - \frac{1}{x}\right), \text{ or } 2\sqrt{-1} \sin. A = \left(x - \frac{1}{x}\right),$$

and similarly, $2\sqrt{-1} . \sin. m A = x^m - \frac{1}{x^m}$;

hence, $\cos. A + \sqrt{-1} \sin. A = x$, $\cos. A - \sqrt{-1} . \sin. A = \frac{1}{x}$,

and, $\cos. mA + \sqrt{-1} . \sin. mA = x^m$, $\cos. mA - \sqrt{-1} \sin. mA = \frac{1}{x^m}$

consequently,

$$(\cos. A + \sqrt{-1} \sin. A)^m = \cos. m A + \sqrt{-1} . \sin. m A,$$

$$\text{and } (\cos. A - \sqrt{-1} . \sin. A)^m = \cos. m A - \sqrt{-1} . \sin. m A.$$

If we expand these expressions, and then add them, we shall have

$$\cos. m A =$$

$$\cos.^m A - \frac{m.(m-1)}{2} \cos.^{m-2} A \times \sin.^2 A + \frac{m(m-1)(m-2)(m-3)}{2.3.4} \times$$

$$\cos.^{m-4} A \times \sin.^4 A - \&c.$$

If we subtract them

$$\sin. m A =$$

$$m . \cos.^{m-1} A \times \sin. A - \frac{m.(m-1)(m-2)}{2.3} \cos.^{m-3} A \times \sin.^3 A + \&c.$$

From the above mode of representing the cosines of multiple arcs we may also deduce, and concisely, the formulæ of Cotes, page 113, &c.* *Theor. Log. Præf. in Harmonia Mensurarum*, and of De Moivre, *Misc. Analyt.* p. 16, &c. thus,

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* The Theorem of Cotes was not announced to the public by its Author, but by the Editor of his Works, Dr. Robert Smith, who informs us, page 113, Preface, that after various conjectures and trials, he extracted it and its meaning from the deceased Author's loose papers "*Revocavi tandem ab interitu Theorema Pulcherrimum.*" M. Lagrange conjectures, and with probability, that Cotes arrived at his Theorem by the way of Vieta's Theorems. See page 54.

In the expression,

$$2 . \cos. mA = x^m + \frac{1}{x^m}, \text{ make } mA = 0;$$

then $x^m + \frac{1}{x^m} = 2 \cos. \theta$, or, $x^{2m} - 2 . \cos. \theta \times x^m + 1 = 0$,

and $x + \frac{1}{x} = 2 . \cos. \frac{\theta}{m}$, or, $x^2 - 2 . \cos. \frac{\theta}{m} x + 1 = 0$.

Now, from $x + \frac{1}{x} = 2 . \cos. \frac{\theta}{m}$, $x^m + \frac{1}{x^m} = 2 \cos. \theta$ was deduced; therefore, if x were deduced from the first expression in terms of $\cos. \frac{\theta}{m}$, or, in other words, if a were the root of the equation, $x^2 - 2 \cos. \frac{\theta}{m} x + 1 = 0$, that same value of x, or root a, substituted in the second expression, $x^m + \frac{1}{x^m} = 2 \cos. \theta$, would make it a true equation, or a would be a root of the equation

$$x^{2m} - 2 \cos. \theta . x^m + 1 = 0.$$

Hence, by the doctrine of equation, $x - a$, is a divisor, both of $x^2 - 2 . \cos. \frac{\theta}{m} x + 1$, and of $x^{2m} - 2 \cos. \theta . x^m + 1$; and similarly, since $\frac{1}{a}$ is the other root of $x^2 - 2 \cos. \frac{\theta}{m} x + 1$, provided a be one root, $x - \frac{1}{a}$ is a divisor both of $x^2 - 2 \cos. \frac{\theta}{m} x + 1$, and of $x^{2m} - 2 \cos. \theta . x^m + 1$; and consequently, $(x-a)\left(x - \frac{1}{a}\right)$, or, $x^2 - 2 . \cos. \frac{\theta}{m} x + 1$ is a divisor of

$$x^{2m} - 2 \cos. \theta . x^m + 1.$$

Now, by Table, p. 16, and by the preceding reasoning, it appears that the arcs

$$2\pi - \theta, \quad 4\pi - \theta, \text{ \&c.} \quad (2n+2)\pi - \theta,$$
$$2\pi + \theta, \quad 4\pi + \theta, \text{ \&c.} \quad 2n\pi + \theta,$$

have the same cosine as the arc θ has: but if, instead of the equation

$$x + \frac{1}{x} = 2\cos.\frac{\theta}{m}, \text{ we assume}$$

$$x + \frac{1}{x} = 2.\cos.\left(\frac{2\pi+\theta}{m}\right) \text{ or } = 2.\cos.\left(\frac{4\pi+\theta}{m}\right) \text{ or } = 2\cos.\left(\frac{2n\pi+\theta}{m}\right)$$

$$\text{or } = 2.\cos.\left(\frac{2\pi-\theta}{m}\right) \text{or} = 2.\cos.\left(\frac{4\pi-\theta}{m}\right) \text{ or } = 2\cos.\left(\frac{(2n+2)\pi-\theta}{m}\right)$$

the resulting expressions will be respectively

$$x^m + \frac{1}{x^m} = \cos.(2\pi+\theta) \text{ or} = 2.\cos.(4\pi+\theta) \text{ or } = 2\cos.(2n\pi+\theta);$$

$$\text{or } = 2.\cos.(2\pi-\theta), \text{ or } = 2\cos.(4\pi-\theta), \text{ or}$$
$$= 2.\cos.(2n+2)\pi - \theta,$$

which expressions, by what has just appeared, (see ll. 1, 2, &c.) are all of equal value.

Hence, of the same expression

$$x^{2m} - 2.\cos.\theta . x^m + 1,$$

$$x^2 - 2\cos.\frac{\theta}{m}x + 1, \; x^2 - 2\cos.\frac{2\pi+\theta}{m}x + 1,$$

$$x^2 - 2.\cos.\left(\frac{2\pi-\theta}{m}\right)x + 1, \; x^2 - 2.\cos.\frac{4\pi+\theta}{m}x + 1, \text{ \&c.}$$

are divisors; in other words, $x^{2m} - 2\cos.\theta . x^m + 1$ may be represented by a product, of which these latter quantities are the factors; accordingly,

$$x^{2m} - 2\cos.\theta . x^m + 1 =$$

$$\left(x^2 - 2.\cos.\frac{\theta}{m}x + 1\right) \times \left(x^2 - 2.\cos.\frac{2\pi+\theta}{m}x + 1\right) \times$$

$$\left(x^2 - 2.\cos.\frac{2\pi-\theta}{m}x + 1\right) \times \text{ \&c.}$$

If we make $\theta = 0$, we have $\cos.\,\theta = 1$, and $\cos.\,\frac{2\pi+\theta}{m} = \cos.\,\frac{2\pi-\theta}{m}$

$$\text{and } x^{2m} - 2x^m + 1, \text{ or } (x^m - 1)^2 =$$

$$(x^2 - 2x + 1).\left(x^2 - 2\cos.\,\frac{2\pi}{m}x + 1\right)^2 \left(x^2 - 2.\cos.\,\frac{4\pi}{m}x + 1\right)^2 \times$$

&c.

and accordingly,

$$x^m - 1 =$$

$$(x - 1).\left(x^2 - 2.\cos.\,\frac{2\pi}{m}x + 1\right).\left(x^2 - \cos.\,\frac{4\pi}{m}x + 1\right).\ \&c.$$

which is the analytical expression of Cotes's Theorem. See *Harmonia Mensurarum*, p. 114, &c. and De Moivre's *Miscellanea Analytica*, p. 17, &c.

We will now proceed to investigate expressions for the powers of the cosine and sine of an arc, in terms of the cosines and sines of multiple arcs, which expressions are highly useful in all Mathematical Investigations connected with Physical Astronomy.

Problem 7. It is required to express the powers of the cosine and sine of an arc, in terms involving the cosines and sines of the multiple arc.

$$2.\cos.\,A = x + \frac{1}{x};$$

$$\therefore 2^n \cos.^n A = \left(x + \frac{1}{x}\right)^n =$$

$$^{*}\,x^n + n x^{n-2} + n.\left(\frac{n-1}{2}\right)x^{n-4} + \&c.\,\frac{n}{x^{n-2}} + \frac{1}{x^n} =$$

(collecting into pairs the terms equidistant from each extremity of the series)

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[*] For the expansion of the Binomial, see Wood's *Algebra*, page 109, first edition; or Vince's *Fluxions*, p. 45; or Woodhouse's *Principles of Analytical Calculation*, page 24, &c.

$$\left(x^n+\frac{1}{x^n}\right)+n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+n.\frac{n-1}{2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)+\&c.$$

∴ by page 48, $2^{n-1}\cos.^n A$

$$=\cos.\,nA+n.\cos.\,(n-2)A+n.\frac{n-1}{2}\cos.\,(n-4)A+\&c.$$

The number of terms is $n+1$, therefore, if n be even, the last term is

$$\frac{n.(n-1)(n-2)\ldots\ldots\left(n-\frac{n}{2}+1\right)}{1.2.3\ldots\ldots\frac{n}{2}}\times\frac{1}{2}\cos.\,(n-n)A.$$

Now,

$$\frac{n.(n-1)(n-2)\ldots\left(n-\frac{n}{2}+1\right)}{1.2.3\ldots\ldots\frac{n}{2}}=\frac{2n(2n-2)\ldots\left(n-\frac{n}{2}+1\right)}{2.4.6\ldots\ldots n}$$

$$=\frac{2n.(2n-2).\&c.}{2.4\ldots\ldots n}\times\frac{1.3.5\ldots\ldots(n-1)}{1.3.5\ldots\ldots(n-1)}=$$

$$2\times2\times2,\ \&c.\left(\text{to }\frac{n}{2}\text{ terms}\right)\times\frac{n.(n-1)(n-2)..\left(\frac{n}{2}+1\right)}{1.2.3,\ \&c.\ldots.(n-1)n}\times1.3.5$$

$$\ldots.(n-1)=2^{\frac{n}{2}}\times\frac{1.3.5\ldots\ldots(n-1)}{1.2.3\ldots\ldots\frac{n}{2}};$$

consequently, since $\cos.\,(n-n)A=1$, the last term $=$

$$2^{\frac{n}{2}-1}\times\frac{1.3.5\ldots\ldots(n-1)}{1.2.3\ldots\ldots\frac{n}{2}}.$$

Hence, as instances of the general form,

$n=2$, $2.\cos.^2 A=\cos.\,2A+1\ldots\ldots\ldots\ldots\ldots\ldots\ldots[e^{II}]$

$n=3$, $2^2.\cos.^3 A=\cos.\,3A+3.\cos.\,A\ldots\ldots\ldots\ldots\ldots\ldots[e^{III}]$

$n=4$, $2^3.\cos.^4 A=\cos.\,4A+4.\cos.\,2A+3\ldots\ldots\ldots\ldots[e^{IV}]$

$n=5$, $2^4.\cos.^5 A=\cos.\,5A+5.\cos.\,3A+10\cos.\,A\ldots.[e^{V}]$

$n=6$, $2^5.\cos.^6 A=\cos.\,6A+6\cos.\,4A+15.\cos.\,2A+10\,[e^{VI}]$

&c. &c.

In order to obtain a general form for $\sin.^n A$, in the form for $\cos.^n A$, substitute instead of A, $\frac{\pi}{2} - A$, then

$$2^{n-1}.\cos.^n \left(\frac{\pi}{2} - A\right) = \cos.\left(n\frac{\pi}{2} - nA\right) + n.\cos.(n-2)\left(\frac{\pi}{2} - A\right)$$
$$+ \frac{n.(n-1)}{2}.\cos.(n-4)\left(\frac{\pi}{2} - A\right) + \&c.$$

Now, let n be even, and of the form $2^{2m}.p$, p being an odd number, or let n be, as it is called, *pariter par;*

then $\frac{n\pi}{2} = 2^{2m-1}p.\pi = 2^{2m-2}p.2\pi$, but $2^{2m-2}.p$ is a whole number, and therefore $\cos.(2^{m-2}p.2\pi)$ is $= 1$.

Again, $(n-2)\frac{\pi}{2} = (2^{2m}p - 2)\frac{\pi}{2} = (2^{2m-1}p - 1)\pi$,

but $(2^{2m-1}p - 1)$ is an odd number, and $\therefore \cos.(2^{2m-1}p - 1)\pi = -1$;

hence, since $\cos.\left(\frac{\pi}{2} - A\right) = \sin.A$, and $\cos.\left(\frac{n\pi}{2} - nA\right) =$

$\cos.\frac{n\pi}{2}\cos.nA = \cos.nA$, [since $\sin.\frac{n\pi}{2} = 0$], &c.

$$2^{n-1}.\sin.^n A =$$
$$\cos.nA - n.\cos.(n-2)A + \frac{n.n-1}{2}.\cos.(n-4)A - \&c.$$

If n be even, but of the form $2p$, or *impariter par* $\frac{n\pi}{2} = 2p\frac{\pi}{2} = p\pi$ and $\cos.p\pi = -1$, and consequently,

since $\cos.(p\pi - nA) = \cos.p\pi.\cos.nA = -\cos.nA$,
$2^{n-1}.\sin.^n A = -\cos.nA + n.\cos.(n-2)A - \&c.$

and, in both these cases, the last term is, as before,

$$\frac{1.3.5\ldots\ldots(n-1).2^{\frac{n}{2}}}{1.2.3\ldots\ldots\ldots\frac{n}{2}} - 1,$$

when n is odd, $\cos.\frac{n\pi}{2} = 0$, and $\cos.\left(\frac{n\pi}{2} - nA\right) = \sin.\frac{n\pi}{2}\sin.nA$

$= \pm$ sin. $n A$, where the upper sign is to be taken, if n be 1, 5, 9, &c. Hence,

$$\pm\ 2^{n-1}.\ \text{sin.}^{n} A. =$$

$$\text{sin.}\ n A\ -\ n.\text{sin.}\ (n\ -\ 2)\ A\ +\ \frac{n.(n-1)}{1.2}\ \text{sin.}\ (n-4)\ A - \text{\&c.}$$

Hence, as instances of the general form,

$n=2,\ 2.\text{sin.}^2 A = -\cos.2 A + 1 \ldots\ldots\ldots\ldots\ldots\ldots[f^{II}]$

$n=3,\ 2^2.\text{sin.}^3 A = -\text{sin.}\ 3 A + 3.\text{sin.}\ A \ldots\ldots\ldots\ldots[f^{III}]$

$n=4,\ 2^3.\text{sin.}^4 A = \cos.4 A - 4.\cos.2 A + 3 \ldots\ldots\ldots[f^{IV}]$

$n=5,\ 2^4.\text{sin.}^5 A = \text{sin.}\ 5 A - 5.\text{sin.}\ 3A + 10\ \text{sin.}\ A \ldots[f^{V}]$

$n=6,\ 2^5.\text{sin.}^6 A = -\cos.\ 6 A + 6 \cos.\ 4A - 15 \cos.\ 2A + 10\,[f^{VI}]$

Problem 8. It is required to express the tangent of twice, thrice, &c. an arc in terms of the tangent of the simple arc.

By Prob. 4, page 35,

$$\tan.\ (A + B) = \frac{\tan.\ A + \tan.\ B}{1 - \tan.\ A.\tan.\ B};$$

consequently,

$$\tan.\ (A+A),\ \text{or,}\ \tan.\ 2 A - \frac{\tan.\ A + \tan.\ A}{1 - \tan.\ A.\tan.\ A} = \frac{2 \tan.\ A}{1 - \tan.^2 A}.$$

Again, $\tan.\ (2 A + A) = \dfrac{\tan.\ 2 A + \tan.\ A}{1 - \tan.\ 2 A.\tan.\ A}$,

and the numerator,

$$\tan.\ 2 A + \tan.\ A = \frac{2 \tan.\ A}{1 - \tan.^2 A} + \tan.\ A = \frac{3 \tan.\ A - \tan.^3 A}{1 - \tan.^2 A},$$

the denominator, which is

$$1 - \tan.\ A.\tan.\ 2 A = 1 - \frac{2.\tan.^2 A}{1 - \tan.^2 A} = \frac{1 - 3.\tan.^2 A}{1 - \tan.^2 A}.$$

Hence

$$\tan.\ 3 A\ \{= \tan.\ (2 A + A)\}\ = \frac{3.\tan.\ A - \tan.^3 A}{1 - 3 \tan.^2 A},$$

and, by a similar method, since $4 A = 3A + A$, and $5 A = 4 A + A$,

$$\tan. 4A = \frac{4 \tan. A - 4 . \tan.^3 A}{1 - 6 . \tan.^2 A + \tan.^4 A},$$

$$\tan. 5A = \frac{5 \tan. A - 10 . \tan.^3 A + \tan.^5 A}{1 - 10 \tan.^2 A + 5 \tan.^4 A}.$$

These expressions for the tangents of multiple arcs may also be derived from those given in p. 37. For instance, since A, B, C, D are equal, t, t', t'', t''' are, and $S_4(t) = 4t$

$$S_4(t t' t'') = \frac{4 \times 3 \times 2}{1 \times 2 \times 3} t^3 = 4 t^3,$$

$$S_4 \ (t t') = \frac{4 \times 3}{1 \times 2} t^2 = 6 t^2$$

$$S_4(t t' t'' t''') = t^4,$$

consequently,

$$\tan. (A + A + A + A) = \tan. 4A = \frac{4t - 4t^3}{1 - 6t^2 + t^4}.$$

as before, in l. 1.

In like manner, we may deduce tan. $5A$ from the expression for tan. $(A + B + C + D + E)$ (see p. 37.)

For,

$$S_5(t) = 5t,$$

$$S_5(t t' t'') = \frac{5 \times 4 \times 3}{1 \times 2 \times 3} t^3 = 10 t^3$$

$$S_5(t t' \ldots . t^{\text{IV}}) = t^3$$

$$S_5(t t') = \frac{5 \times 4}{1 \times 2} t^2 = 10 t^5$$

$$S_5(t t' t'' t''') = \frac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4} t^4 = 5 t^4;$$

$$\therefore \tan. 5A = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

as before.

It is not, perhaps, necessary to multiply farther trigonometrical formulæ; such as are chiefly useful, and usually occur in investigation, have been given; and the Student, who thoroughly apprehends the principle and mode of their deduction, will be able, by his own dexterity, to deduce others.

A sufficient number of formulæ having been given, it may now be thought proper to proceed to their application; and, the first object of their application seems naturally to be that for which the science of Trigonometry was originally invented; namely, the Solution of Rectilinear Triangles. Now, this solution consists of two parts; first, it is necessary to express the relations of the sides and angles of triangles by Trigonometrical symbols; and, secondly, to afford the means of arithmetically computing, in specific instances, the values of such symbols. For instance, if two sides, a, b, and an angle A of a rectilinear triangle should be given, the value of the angle B (see p. 26.) would be truly expressed by

$$\sin. B = \sin. A \times \frac{b}{a}.$$

But this is an algebraical value; in order to obtain a practical result, we must be able, when b and a are expounded by numbers, and A by degrees, minutes, &c. to express B in degrees, minutes, &c. we must, therefore, possess the means of assigning sin. A from a given value of A, and also of assigning B from a resulting value of sin. B. These means, in practice, are afforded by Trigonometrical Tables, and their formation, or, what technically is called the construction of the *Trigonometrical Canon*, is an easy consequence from the preceding results.

We will, in the next Chapter, proceed to the construction of this Canon, which may be viewed either as a distinct application of the preceding formulæ, or as a preparatory step to their application in the solution of rectilinear triangles.

CHAP. IV.

On the Construction of the Trigonometrical Canon.—Methods of computing the Sine of 1′.—The Sines and Cosines of successive Arcs.—Formulæ of Verification, or Methods of examining the Accuracy of Computed Tables.

Problem 9. It is required to find a numerical value of the sine of 1 minute; the circle being divided into 360 × 60 or 21600 minutes, and its radius being 1.

By Prob. 5, $\cos. A = \sqrt{\{\frac{1}{2}(1 + \cos. 2A)\}}$ [*a*], and by Cor. 2, Prob. 5, if $2A = 60^0$, $\cos. 2A = \frac{1}{2} = .5$,

consequently, $\cos. 30^0 = \sqrt{\{\frac{1}{2}(1 + \frac{1}{2})\}} = .8660254$;

substitute this value into the form [*a*], and we shall have cos. 15°: and, by a repetition of the operation, successively, the cosines of $\frac{60^0}{2^3}$, $\frac{60^0}{2^4}$, $\frac{60^0}{2^5}$, $\frac{60^0}{2^{12}}$: so that, if c, c^I, c^{II}, &c. stand for the successive cosines, the operation may be thus exhibited:

$$\cos.\frac{60^0}{2}, \text{ or, } \cos. 30^0 = \sqrt{\left\{\frac{1}{2}\left(1+\frac{1}{2}\right)\right\}} = c^I = .8660254, \text{ \&c.}$$

$$\cos.\frac{60^0}{2^2}, \text{ or, } \cos. 15^0 = \sqrt{\left(\frac{1}{2}(1+c^I)\right)} = c^{II} = .9659258, \text{ \&c.}$$

$$\cos.\frac{60^0}{2^3}, \text{ or, } \cos. 7^0\,30' = \sqrt{\left(\frac{1}{2}(1+c^{II})\right)} = c^{III} = .9914449, \text{ \&c.}$$

$$\cos.\frac{60^0}{2^4}, \text{ or, } \cos. 3^0\,45' = \sqrt{\left(\frac{1}{2}(1+c^{III})\right)} = c^{IV} = .9978589, \text{ \&c.}$$

&c. &c.

*cos. $\frac{60^\circ}{2^{12}}$, or, cos. $52'' \; 44'''.3^{IV}.45^{V} = \sqrt{\left(\frac{1}{2}(1+c^{XI})\right)} =$

.99999996732.

From this value of the cosine, the sine = .000255663462.

In order to find the sin. 1′, we must compute on this principle, namely, that the sines of very small arcs are to one another as the arcs themselves †; and then, since the arcs $52''.44'''.3^{IV}.45^{V}$, and 1′ are to one another as $\frac{60}{2^{12}} : \frac{60}{60 \times 60}$,

---

* The values of $\frac{60^\circ}{2^{12}}$ may be shortly obtained by the following mode of decomposing numbers:

$$\frac{60^\circ}{2^{12}} = \frac{60.60.60}{2^{12}} \text{(the units being seconds)} = \frac{(64-4)(64-4)(64-4)}{2^{12}}$$

$$= \frac{(2^6-2^2)^3}{2^{12}} = \frac{(2^4-1)^3}{2^6} = \left(\frac{2^4-1}{2^2}\right)^3 = \left(2^2 - \frac{1}{2^2}\right)^3 = 2^6 - 3.2^2 + \frac{3}{2^2} - \frac{1}{2^6};$$

$$\text{but } 2^6 + \frac{3}{2^2} = 64''.45''',$$

$$3.2^2 + \frac{1}{2^6} = 12''.0.56^{IV}.15^{V},$$

$$\text{subtracting, } \frac{60^\circ}{2^{12}} = 52''.44'''.3^{IV}.45^{V}.$$

† If from the two extremities of an arc there be drawn two lines touching the arc and meeting each other, such lines will be equal. By the principle assumed by Archimedes, the arc is < sum of tangents > chord joining the two ends of the arc; consequently $\frac{1}{2}$ arc < tangent > $\frac{1}{2}$ chord or > sine of ½ arc: and therefore tan. $\frac{A}{2}$ − sin. $\frac{A}{2} > \frac{A}{2}$ − sin. $\frac{A}{2}$: hence, if, in the instance given in the text, we find the difference between the tangent and sine of 1′ (computing tan. 1′ from $\frac{\text{sin. } 1'}{\text{cos. } 1'}$), it will be found to be .000290888216 − .000290888204 = .000000000012: consequently, the arc of 1′ differs from its sine by a quantity less than .000000000012: so that, it is plain, the principle of very small arcs varying as their sines, is very little remote from the truth; or, rather, if assumed will entail on the computation a very small numerical error.

or, as 3600 : 4096,

we have, $\sin. 1' = .000255663462 \times \frac{4096}{3600} = .000290888204.$

This method of computing the sine of 1′, although very operose, is of no great difficulty; for, it requires only the knowledge of the simplest arithmetical operations. But, even if we avail ourselves of the formulæ and inventions of the Analytic art, the computation of sin. 1′ cannot, in any case, be very expeditiously effected; more expeditiously, however, than by the preceding method.

If we employ the expression for sin. A which was given, in page 44, we may successively deduce the sines of 30°, 15°, 7° 30′, &c. by operations analogous to those already given; thus, since $\sin. A = \frac{1}{2}\sqrt{(1+\sin. 2A)} - \frac{1}{2}\sqrt{(1-\sin. 2A)}$, and $\sin. 30^0 = \frac{1}{2}$.

$$\begin{array}{r|l}
A=30^\circ & \sin. 30^\circ \quad = \frac{1}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = .5 \\
A=15^\circ & \sin. 15^\circ \quad = \frac{1}{2}\sqrt{(1+\frac{1}{2})} - \frac{1}{2}\sqrt{(1-\frac{1}{2})} = s' = .258819 \\
A=7^\circ\, 30' & \sin. 7^\circ\, 30' = \frac{1}{2}\sqrt{(1+s')} - \frac{1}{2}\sqrt{(1-s')} = s'' = .1305262 \\
A=3^\circ\, 45' & \sin. 3^\circ\, 45' = \frac{1}{2}\sqrt{(1+s'')} - \frac{1}{2}\sqrt{(1-s'')} = s''' = .0654031
\end{array}$$

and so on we may compute till we obtain the $\sin. \frac{30^0}{2^{11}}$.

The preceding computation was made to begin from $A=30^0$, because the sin. 30° is known; but we might have begun from any other arc, the sine of which is known: thus, if we take $2A = 18^0$, the sine of $18^0 = \frac{1}{2}$ chord 36°, but the chord $36^0 =$ the side of a regular decagon inscribed in a circle, $= BD$ the base of the isosceles triangle described in the 10th Proposition 4th Book of Euclid, * $= \frac{\sqrt{5}-1}{2}$: hence, $\sin. 18^0 = \frac{\sqrt{5}-1}{4}$, and substituting this value for sin. $2A$ in the above form, we have

$$\begin{array}{ll}
\sin. 9^\circ & = \frac{1}{4}\sqrt{(3+\sqrt{5})} - \frac{1}{4}\sqrt{(5-\sqrt{5})} = s = .156434 \\
\sin. 4^\circ\, 30' & = \frac{1}{2}\sqrt{(1+s)} \quad - \frac{1}{2}\sqrt{(1-s)} \quad = s' = .0784591 \\
\sin. 2^\circ\, 15' & = \frac{1}{2}\sqrt{(1+s')} \quad - \frac{1}{2}\sqrt{(1-s')} \quad = s'' = .0392598 \\
\&c. = \&c. &
\end{array}$$

---

* $BAD + 2ABD = 180^\circ$, but $ABD = 2BAD$; $\therefore 5BAD = 180^\circ$; $\therefore 10BAD = 360^\circ$ and $BAD = 36^\circ$: again, by the Prop. $AB \cdot BC = AC^2 = BD^2$, consequently, if $BD = x$, $AB = 1$, $1 \cdot (1-x) = x^2$, and $x = \frac{\sqrt{5}-1}{2}$.

Having thus computed the sine of a small arc that is nearly 1′, the method of determining the sin. 1′ is, in principle, precisely the same as in Prob. 9. There is, however, a third method, considerably different in its principle, which finds the sine of 1′ by the quinquesection and trisection of an arc. It may be thus explained,

By the form $[s^{v}]$, page 47,

$$\sin. 5A = 5 \sin. A - 20 \sin.^3 A + 16 \sin.^5 A.$$

Let $5A = 30^0$ then $\sin. 5A = \frac{1}{2}$ and $A = 6^0$; let $2 \sin. A = x$, then

$$1 = 5x - 5x^3 = x^5.$$

By approximation, find the value of x: thus, suppose a to be a near value, and $a + v$ to be the true, then

$$1 = 5(a+v) - 5(a^3 + 3a^2 v) + a^5 + 5a^4 v,$$

neglecting the terms that involve v^2, v^3, &c. consequently,

$$v = \frac{1 - 5a + 5a^3 - a^5}{5(1 - 3a^2 + a^4)}.$$

Now, since $\sin. 5A = \frac{1}{2}$, assume, as a first approximate value of x, $a = \frac{2}{10} = .2$: substitute in the expression for v this value, and find the resulting value of v; it will appear to be $= .009$, the corrected value of a then, or $a+v$, is $.209$; with this, find a new value of v, and another corrected value of a, and repeat the operation till x is found exact to a certain number of decimals, seven for instance; in which case

$$x = .2090569, \text{ and consequently } \sin. A = .1045285.$$

Having thus obtained the sine of 6^0, in the form $[s''']$ page 47, that is, in $\sin. 3A = 3 \sin. A - 4 \sin.^3 A$, put $3A = 6^0$, and $2 \sin. A = x$, then the equation becomes $.2090569 = 3x - x^3$. Find, as before, by the method and formula of approximation, a value of x, which, to seven places of figures, will be $.0697989$, consequently x or $2 \sin. 2^0 = .0697989$, and $\sin. 2^0 = .0346995$.

In order to find sin. 1^0, take the form (p. 42,) $\sin. 2A = 2 \sin. A.$

cos. A, then $\sin.^2 2A = 4 . \sin.^2 A - 4 . \sin.^4 A$: substitute for $\sin.^2 2A$, or $\sin.^2 2^\circ$, its value, and by the solution of a quadratic equation find $\sin.^2 A$, and then, sin. A, or sin. 1°, the value of which to seven places of figures is .0174524. Repeat the operation, and we have $\sin. \frac{1^\circ}{2}$, or sin. 30′, the value of which is .0087265.

By this method then we have descended from the sin. 30° to the sin. 1° and sin. 30′; and consequently, by like operations, we can descend from sin. 30′ to sin. 1′ and sin. 30″: and by this method, which is however extremely operose, we are able to find the sin. 1′ without a proportion, and accordingly to avoid the use of a principle, which some may think doubtful; namely, that the sines of small arcs are to one another as the arcs themselves.

The above method is, in fact, the same as that which is given with all its detail, at page 451, &c. in the sixth Volume of the *Scriptores Logarithmici*, edited by Baron Maseres. It is plain, however, that there is no necessity for beginning the computation from an arc of 30°; we may make it begin from any arc, the sine of which is known: for instance, by the form, page 67,

$$\sin. 9^\circ = \tfrac{1}{4}\sqrt{(3+\sqrt{5})} - \tfrac{1}{4}\sqrt{(5-\sqrt{5})} = .156434;$$

since, therefore, $\sin. 9^\circ = \sin. (3 . 3^\circ) = 3 \sin. 3^\circ - 4 \sin.^3 3^\circ$, solve, as before, by approximation, the equation

$$.156434 = 3x - 4x^3,$$

and the result gives x the sin. 3°. Again, solve a similar equation by the same mode and formula, and the result gives sin. 1°. And many like methods will suggest themselves to the mind of the intelligent Student.

We shall now proceed to the second part of the construction of Trigonometrical Tables, the object of which will be understood from the succeeding Problem.

PROBLEM 10. It is required from the sin. 30″ and sin. 1′, to compute the sines of 2, 3, 4, &c. minutes, and also the sines of 1, 2, 3, &c. degrees.

By the form [*a*], page 31,

$$\sin.(A+B) = 2\sin.A.\cos.B - \sin.(A-B)$$

$$= 2\sin.A\left(1 - 2\sin.^2\frac{B}{2}\right) - \sin.(A-B)$$

$$= \sin.A + \{\sin.A - \sin.(A-B)\} - 4\sin.A\sin.^2\frac{B}{2}.$$

If $B = 1^o$, then

$$\sin.(A+1^o) = \sin.A + \sin.A - \sin.(A-1^o) - 4\sin.A\sin.^2 30'$$

which is Delambre's formula.

Let $B = 1'$ and let A successively equal 1′, 2′, 3′, &c. then

$$\sin.2' = \sin.1' + (\sin.1' - \sin.0) - 4\sin.1'.(\sin.30'')^2$$
$$\sin.3' = \sin.2' + (\sin.2' - \sin.1') - 4\sin.2'.(\sin.30'')^2$$
$$\sin.4' = \sin.3' + (\sin.3' - \sin.2') - 4\sin.3'.(\sin.30'')^2$$
$$\sin.5' = \sin.4' + (\sin.4' - \sin.3') - 4\sin.4'.(\sin.30'')^2,$$
&c. = &c.

and thus may the sines of all succeeding arcs be computed, by a process not very tedious, since the only part of it at all long is the multiplication of sin. 2′, sin. 3′, &c. by the constant factor $(2\sin.30'')^2$, which is the square of the chord of 1′.

In the above form substitute, instead of B, 1°, and, instead of A, successively 1, 2, 3, &c. degrees; then

$$\sin.2^o = \sin.1^o + (\sin.1^o - \sin.0) - 4\sin.1^o(\sin.30')^2$$
$$\sin.3^o = \sin.2^o + (\sin.2^o - \sin.1^o) - 4\sin.2^o(\sin.30')^2,$$
&c.

and so on for the sines of all succeeding arcs.

In order to compute the sines of arcs composed of degrees and minutes; arcs, for instance, such as 3° 2′, 3° 3′, substitute for B, 1′, and for A successively 3° 1′, 3° 2′, 3° 3′, &c. then

$$\sin.3^o\,2' = \sin.3^o\,1' + (\sin.3^o\,1' - \sin.3^o) - 4\sin.3^o\,1'(\sin.30'')^2$$
$$\sin.3^o\,3' = \sin.3^o\,2' + (\sin.3^o\,2' - \sin.3^o\,1') - 4\sin.3^o\,2'(\sin.30'')^2$$
&c.

or, if we wish to compute for every ten minutes, put $B = 10'$, and for A write successively $A + 10'$, $A + 20'$, $A + 30'$, &c. thus, if $A = 7^\circ$,

$$\text{sin. } 7^\circ\, 20' = \text{sin. } 7^\circ\, 10' + (\text{sin. } 7^\circ\, 10' - \text{sin. } 7^\circ) - 4 \text{ sin. } 7^\circ\, 10'.(\text{sin. } 5')^2$$
$$\text{sin. } 7^\circ\, 30' = \text{sin. } 7^\circ\, 20' + (\text{sin. } 7^\circ\, 20' - \text{sin. } 7^\circ\, 10') - 4 \text{ sin. } 7^\circ\, 20'.(\text{sin. } 5')^2,$$
&c.

By the preceding methods we are enabled regularly to compute the sines of all arcs from $1'$ or $1''$ up to 90°; but, when the arcs exceed 60°, the application of the Trigonometrical formula, page 31, renders the arithmetical computation more simple and concise: thus, since

$$\text{sin. } (B + A) = \text{sin. } (B - A) + 2 . \text{cos. } B . \text{sin. } A.$$

Let $B = 60^\circ$, then cos. $B = \text{cos.} \, 60^\circ = \frac{1}{2}$, consequently,

$$\text{sin. } (60^\circ + A) = \text{sin. } A + \text{sin. } (60^\circ - A).$$

(see form 15 of Table, p. 39.)

Hence, instead of the preceding, we may use this latter method, and compute the sines of all arcs exceeding 60°, by the simple addition of the sines of arcs previously computed: for instance,

$$\text{sin. } 63^\circ\, 5' = \text{sin. } 3^\circ\, 5' + 56^\circ\, 55'$$
and, since sin. $3^\circ\, 5' = .0537883$, and sin. $56^\circ\, 55' = .8378775$
the sin. $63^\circ\, 5' = .891665$.

The sine of all arcs from 0 to 90° being computed, the cosines of all the arcs of the quadrant are known; since cos. $A =$ sin. $(90^\circ - A)$; for instance, cos. $63^\circ\, 15'\, 7'' =$ sin. $26^\circ\, 44'\, 53''$ cos. $13^\circ\, 47' =$ sin. $76^\circ\, 13'$, &c.

The sines and cosines being computed, the tangents may be computed from this expression, tan. $A = \dfrac{\text{sin. } A}{\text{cos. } A}$, and the co-tangents from co-tan. $A = \dfrac{\text{cos. } A}{\text{sin. } A}$.

When the tangents of arcs up to 45° have been computed, the Trigonometrical formulæ previously given may be conveniently used in computing the tangents of arcs that lie between 45° and 90°: thus, by Prob. 4, page 35,

$$\text{tan.}\,(A+B) = \frac{\text{tan.}\,A + \text{tan.}\,B}{1-\text{tan.}\,A.\text{tan.}\,B} = [\text{if } B = 45^\circ]\,\frac{1 + \text{tan.}\,A}{1 - \text{tan.}\,A};$$

hence, as instances, putting $A = 1^\circ, 2^\circ, 3^\circ$, &c.

$$\text{tan.}\,(45^\circ + 1^\circ), \text{ or, tan.}\,46^\circ = \frac{1 + \text{tan.}\,1^\circ}{1 - \text{tan.}\,1^\circ},$$

$$\text{tan.}\,(45^\circ + 2^\circ), \text{ or, tan.}\,47^\circ = \frac{1 + \text{tan.}\,2^\circ}{1 - \text{tan.}\,2^\circ},$$

&c. &c.

or, we may thus avoid the fractional form,

$$\text{tan.}\,(45^\circ + A) = \frac{1+\text{tan.}\,A}{1-\text{tan.}\,A} \text{ and tan.}\,(45^\circ - A) = \frac{1 - \text{tan.}\,A}{1 + \text{tan.}\,A};$$

$$\therefore\ \text{tan.}\,(45^\circ + A) - \text{tan.}\,(45^\circ - A) = \frac{(1 + \text{tan.}\,A)^2 - (1 - \text{tan.}\,A)^2}{1 - \text{tan.}^2\,A}$$

$$= \frac{4\,.\,\text{tan.}\,A}{1-\text{tan.}^2\,A};$$

but, by the form of page 62, $\text{tan.}\,2\,A = \frac{2\ \text{tan.}\,A}{1 - \text{tan.}^2\,A}$.

Hence, $\text{tan.}\,(45^\circ + A) = 2\ \text{tan.}\,2\,A + \text{tan.}\,(45^\circ - A)$, consequently,

$$\text{tan.}\,(45^\circ + 1^\circ), \text{ or, tan.}\,46^\circ = 2\ \text{tan.}\,2^\circ + \text{tan.}\,44^\circ,$$
$$\text{tan.}\,(45^\circ + 2^\circ), \text{ or, tan.}\,47^\circ = 2\ \text{tan.}\,4^\circ + \text{tan.}\,43^\circ,$$
&c.

By these formulæ and methods may the sines, tangents, &c. of arcs be computed. If we attend, however, to the history of the construction of Trigonometrical Tables, we shall find that all Tables have not been computed exactly by the same formulæ and methods: modern Tables, from the improved state of analytic science, have been computed by the most certain and expeditious methods. In the immense *Tables du Cadastre*, formed at the expense of the French Government, the sines of arcs are computed regularly by successive addition, according to the formulæ given in page 70; but, in such a construction, an error committed in the sine of an inferior arc would, it is plain, entail errors on the sines of all succeeding arcs. Hence is created the necessity of some check on the computist, and of some independent mode of examining the accuracy of the computation. For this purpose, formulæ, such as those given in pages 43, 67, derived immediately from established properties, are employed; if the numerical results from these formulæ agree with the results obtained by the regular process of computation, then, it is almost a certain conclusion that the latter process has been rightly conducted.

As there is, in these formulæ, called *Formulæ of Verification*, besides their practical utility, something curious, several are subjoined and proved.

[1] $\sin. 30^0 = \frac{1}{2}$; $\sin. 45^0 = \frac{1}{\sqrt{2}}$; $\sin. 60^0 = \frac{\sqrt{3}}{2}$.

[2] $\sin. 18^0 = \frac{\sqrt{5}-1}{4}$, by the Note to p. 67, or, it may be be independently proved thus:

$\sin. 36^0 = \cos. 54^0$, or, $\sin. (2 . 18^\circ) = \cos. (3 . 18^\circ)$. Let $x = \cos. 18^\circ$; then by form [c'''], page 45, $\cos. (3 . 18^\circ) = 4x^3 - 3x$, and by form, p. 42, $\sin. (2 . 18^\circ) = 2x\sqrt{(1-x^2)} = \therefore 4x^3 - 3x$, or, $2\sqrt{(1-x^2)} = 4x^2 - 3$.

This equation, cleared of radicals and reduced, is

$$16x^4 - 20x^2 + 5 = 0;$$

whence $x^2 = \frac{5+\sqrt{5}}{8}$ and $\therefore 1 - x^2 = \frac{3-\sqrt{5}}{8}$ and $\sqrt{1-x^2}$, or $\sin. 18^0 = \frac{\sqrt{5}-1}{4}$ which is also cos. 72°.

[3] $\sin. 9^0$, or, $\cos. 81^0 = \frac{1}{4}\sqrt{(\sqrt{5}+3)} - \frac{1}{4}\sqrt{(5-\sqrt{5})}$, by the form, of p. 43, on substituting for sin. 18° its value,

$\cos. 9^\circ$, or, $\sin. 81^\circ = \frac{1}{4}\sqrt{(3+\sqrt{5})} + \frac{1}{4}\sqrt{(5-\sqrt{5})}$, by the same form.

[4] $\sin. 27^\circ$, or, $\cos. 63^\circ = \frac{1}{4}\sqrt{(5+\sqrt{5})} - \frac{1}{4}\sqrt{(3-\sqrt{5})}$
$\cos. 27^\circ$, or, $\sin. 63^\circ = \frac{1}{4}\sqrt{(5+\sqrt{5})} + \frac{1}{4}\sqrt{(3-\sqrt{5})}$

for, in the preceding proof of the arith^l. value of sin. 18°, $(\cos. 18^\circ)^2 = \frac{5+\sqrt{5}}{8}$ and $(\sin. 18^\circ)^2 = \frac{3-\sqrt{5}}{8}$; $\therefore \cos. 2 . 18^\circ$, or, $\cos. 36^\circ =$

$(\text{cos. } 18^\circ)^2 - (\text{sin. } 18^\circ)^2 = \frac{2+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4} = \text{sin. } 54^\circ$, since cos. $36^\circ = \text{sin. } 54^\circ$,

substitute this value of the sin. 54° in the forms of page 43, for sin. $2A$, and there will result the above values for the sine and cosine of 27°.

$$\text{Since, cos. } 36^\circ = \frac{\sqrt{5}+1}{4}, \text{ sin. } 36^\circ = \frac{\sqrt{(10-2\sqrt{5})}}{4}.$$

The subjoined Table contains the sines of arcs from 0° to 90° that differ by 9°, or, in the French division of the circle, that differ by 10 degrees.

French Scale.	English Scale.	
sin. 10°, or, cos. 90°	sin. 9°, or, cos. 81°	$=\frac{1}{4}\sqrt{(3+\sqrt{5})}-\frac{1}{4}\sqrt{(5-\sqrt{5})}$
sin. 20, or, cos. 80	sin. 18, or, cos. 72	$=\frac{1}{4}(\sqrt{5}-1)$
sin. 30, or, cos. 70	sin. 27, or, cos. 63	$=\frac{1}{4}\sqrt{(5+\sqrt{5})}-\frac{1}{4}\sqrt{(3-\sqrt{5})}$
sin. 40, or, cos. 60	sin. 36, or, cos. 54	$=\frac{1}{4}\sqrt{(10-2\sqrt{5})}$
sin. 50, or, cos. 50	sin. 45, or, cos. 45	$=\frac{1}{\sqrt{2}}$
sin. 60, or, cos. 40	sin. 54, or, cos. 36	$=\frac{1}{4}(\sqrt{5}+1)$
sin. 70, or, cos. 30	sin. 63, or, cos. 27	$=\frac{1}{4}\sqrt{(5+\sqrt{5})}+\frac{1}{4}\sqrt{(3-\sqrt{5})}$
sin. 80, or, cos. 20	sin. 72, or, cos. 18	$=\frac{1}{4}\sqrt{(10+2\sqrt{5})}$
*sin. 90, or, cos. 10	sin. 81, or, cos. 9	$=\frac{1}{4}\sqrt{(3+\sqrt{5})}+\frac{1}{4}\sqrt{(5-\sqrt{5})}$.

But the most general formula of verification is this, which is to be found in Euler's *Analysis Infinitorum*, page 201, vol. I. (Lausanne, 1748).

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* There is a large Table of this kind in Cagnoli's *Trigonometry*, p. 58, &c. edit. 2. And, it is easy to see how additional formulæ may be obtained: for instance,

$$\text{sin. } 15^\circ = \text{sin. } (45^\circ - 30^\circ) = \text{sin. } 45^\circ\,(\text{cos. } 30^\circ - \text{sin. } 30^\circ)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \frac{1}{2\sqrt{2}}(\sqrt{3}-1).$$

$$\text{Similarly, sin. } 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

And these and like expressions, besides that utility which is pointed out in the text, may have a farther one, in the theory of Polygonals.

[5] $\sin. A + \sin. (36^\circ - A) + \sin. (72^\circ + A) = \sin. (36^\circ + A) + \sin. (72^\circ - A)$.

In order to prove this formula, we will use the numerical values of the cosines of 36° and 72°.

$$\sin. (36^\circ + A) - \sin. (36^\circ - A) = 2 . \cos. 36^\circ . \sin. A = \frac{\sqrt{5}+1}{2} . \sin. A$$

and $\sin. (72^\circ + A) - \sin. (72^\circ - A) = 2 . \cos. 72^\circ . \sin. A = \frac{\sqrt{5}-1}{2} . \sin. A$;

subtract the latter equation from the former, and

then we have, $\sin. (36^\circ + A) + \sin. (72^\circ - A) - \sin. (72^\circ + A) - \sin. (36^\circ - A) = \sin. A$, which transposed is the equation [5].

If, in the above equation, we substitute $90^\circ - A$ instead of A, there will result

$$\sin. (90^\circ - A) + \sin. (A - 54^\circ) + \sin. (18^\circ + A) = \sin. (54^\circ + A) - \sin. (18^\circ - A), \text{ or}$$

$$\sin. (90^\circ - A) = \sin. (54^\circ + A) + \sin. (54^\circ - A) - \sin. (18^\circ + A) - \sin. (18^\circ - A),$$

which is Legendre's mode of expressing the equation. But, it is plain from the mode by which the latter has been deduced, there is no real difference between the two formulæ, and, with regard to their application, it is quite indifferent, whether we adopt Euler's or Legendre's.

In using these formulæ, different values must be substituted for A, thus: in Euler's,

if $A = 9^\circ$, then, $\sin. 9^\circ + \sin. 27^\circ + \sin. 81^\circ = \sin. 45^\circ + \sin. 63^\circ$;
or, $\sin. 9^\circ + \sin. (3 . 9^\circ) + \sin. (9 . 9^\circ) = \sin. (5 . 9^\circ) + \sin. (7 . 9^\circ)$.

If in Legendre's formula we make $A = 81^\circ$; then

$$\sin. 9^\circ = \sin. 45^\circ - \sin. 27^\circ - \sin. 81^\circ + \sin. 63^\circ;$$
$$\text{or } \sin. 9^\circ + \sin. 27^\circ + \sin. 81^\circ - \sin. 45^\circ + \sin. 63^\circ,$$

the same as before; which proves what we have just asserted, (l. 11, &c.)

Again, if $A = 18^\circ$, then, by the formula [5]

$\sin. 18^\circ + \sin. (36^\circ - 18^\circ) + \sin. 90^\circ = \sin. (36^\circ + 18^\circ) + \sin. (72^\circ - 18^\circ)$;

or, $2 \sin. 18° + 1 = 2 \sin. 54°$.

If $A = 10°$, then $\sin. 10° + \sin. 26° + \sin. 82° = \sin. 46° + \sin. 62°$.

EXAMPLES.

Take Sherwin's Tables, in which the natural sines of arcs are inserted: from these it appears, that

sin. 9° = 1564345	
sin. 27 = 4539905	sin. 45° = 7071068
sin. 81 = 9876883	sin. 63 = 8910065
15981133	15981133
sin. 10° = 1736482	
sin. 26 = 4383711	sin. 46° = 7193398
sin. 82 = 9902681	sin. 62 = 8829476
16022874	16022874

Here the numerical results of the two sums exactly agreeing, as they ought to do according to the formula, we may conclude, almost with certainty, that in Sherwin's Tables the sines of 9, 27, 81, 45, 63, 10, 26, 46, 62 degrees are rightly computed.

These formulæ are most convenient for practice; but if, from the solution of an equation of three dimensions, as (s^{III}) p. 47, or, from that of an equation of five dimensions, as (s^{V}), the value of $\sin. \frac{A}{3}$, or of $\sin. \frac{A}{5}$ be computed, such value would become a means of ascertaining the accuracy of Trigonometrical Tables.

In 1610, Pitiscus published great Trigonometrical Tables, inserted in his *Mathematicus Thesaurus*, and, from the account of this Work, given in the Berlin Memoirs of 1786, page 24, it appears, that formulæ, which, in fact, are formulæ of verification, were employed by that mathematician: thus, in order to ascertain whether the chord of 30° had been rightly computed, he substitutes in the equation,

*$4x^2 - x^4 = c^2$ (x = chord of $30°$, c = chord $60°$)

instead of x the computed value, and he finds the resulting value of c to be 1000; as it ought to be, since the chord $60° =$ radius: and accordingly he concludes x to have been rightly computed. Again, in order to ascertain that the chord of $10°$ is rightly computed, he substitutes its computed value in an equation, such as chord $30° = 3$ chord $10° - (\text{chord } 10)^3$ [see form [s'''] page 47] and the chord $30°$ thence computed ought to agree with its value (x) previously ascertained to be right; and he pursues a similar course, in order to verify the computation of the chord of the fifth part of an arc.

Previously to quitting this part of our subject, we wish to employ the arithmetical values of the sines of 18° and of 36°, which have been just deduced, in proving that which may be announced as a geometrical property.

By the form p. 74, $\cos. 36° = \frac{\sqrt{5}+1}{4}$, and $\therefore \sin.^2 36° = \frac{5-\sqrt{5}}{8}$;

consequently, $4 \sin.^2 36° - 1 = \frac{3-\sqrt{5}}{2} = 4 \sin.^2 18°$, see page 73,

or $(\text{chord } 72°)^2 - 1 = (\text{chord } 36°)^2$, but the chords of 72° and 36° are respectively the sides of an equilateral pentagon and decagon, inscribed in a circle. Hence, the square of the side of an equilateral pentagon inscribed in a circle, is equal to the square of the radius plus the square of the side of an equilateral decagon inscribed in the same circle.

Having now obtained methods of arithmetically computing the sines, cosines, &c. of angles, when the angles are expounded by a specific number of degrees, minutes, &c., we may proceed to apply our formulæ to express the various relations that subsist between the sides and angles of rectilinear triangles.

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* Pitiscus's notation is like Vieta's, given in page 54; the form of his equation is $4q - 1bq =$ square of chord of twice the arc. Another account of this Work of Pitiscus, and of similar Works, published about the same time, and now very rare, is given in the 5th Volume of the *Memoirs of the Institute*.

CHAP. V.

On the Resolution of the Cases of Rectilinear Triangles—1st, When the Triangles are Right-angled.—2d, When Oblique.—Reasons for introducing different Solutions of the same Case. Examples, &c.

In a triangle there are 3 sides and 3 angles: any three of these being given, the remaining may be obtained. There is one exception to this, which takes place when the three quantities given are the 3 angles. The reason of the exception is this: take any triangle, then, externally or internally, other triangles may be formed with sides parallel to the sides of the proposed triangle, which triangles shall have the same angles, but greater sides or less sides: the magnitudes of the sides therefore are independent of the angles, and consequently cannot be determined from them.

We will begin with the solutions of the *cases of right-angled triangles.*

1st Case of right-angled triangles, in which two sides are given.

Here, besides the right angle *C, a, b,* are given, and *c, A, B* are required.

Solution.

c determined.

1st. $c=\sqrt{(a^2+b^2)}$ Euclid 47. Book 1.

A and *B* determined.

2d. $\frac{\sin. A}{\sin. B}=\frac{a}{b}$ by Cor. 1. to Prob. 2.

but $A+B=\frac{\pi}{2}$ or 90° $\therefore \sin. B=\cos. A$

$\therefore \frac{\sin. A}{\cos. A}$, or $\tan. A=\frac{a}{b}$ (rad. = 1), or

$\tan. A=r\,.\,\frac{a}{b}$ and expressed in $\log^{ms}$.

$\log. \tan. A=\log. r+\log. a-\log. b$

$$B=90^\circ-A$$

A being determined, *c* may: for

$$c=\frac{b.\sec. A}{r}=\frac{br}{\cos. A}$$

$\therefore \log. c=\log. r+\log. b-\log. \cos. A.$

Example.

$a=43$; $\therefore a^2=1849$

$b=55$; $\therefore b^2=3025$

$\therefore c^2=4874$

and $c=69{,}81$, &c.

Computation.

r the tabular radius $=10^{10}$

log. r = ...10

log. 43=......1.6334685

$\overline{11.6334685}$

log. 55=......1.7403627

$\therefore$ log. tan. $A=\overline{9.8931058}$

$\therefore A=38^\circ\ 1'\ 8''$

and $B=51\ \ 58\ \ 52$

Again, log. $r=10$

log. 55=1.7403627

$\overline{11.7403627}$

log. cos. $38^\circ\ 1'\ 8''$=9.8964202

log. $c=\overline{1.8439425}$

$\therefore c=69, 81$, &c. as before.

2d Case, in which the hypothenuse and one of the acute angles are given.

Here c, B, and $C = 90°$, are given, and a, b, A required.

Solution.

b determined.

$$\frac{\sin. B}{\sin. C} = \frac{b}{c} \text{ or } \frac{\sin. B}{1} = \frac{b}{c}$$

$$\therefore b = c . \sin. B = c . \frac{\sin. B}{r} \text{ (rad.} = r)$$

In logarithms,

$$\log. b = \log. c + \log. \sin. B - \log. r.$$

Example.

$c = 361.4$, $B = 41° \ 12'$

Computation.

$\log. 361.4 = 2.5579881$
$\log. \sin. 41° \ 12' = 9.8186807$

$\log. b = 2.3766688$
$\therefore b = 238.05$

$$A = 90° - B = 48° \ 48'; \therefore \frac{\sin. 48° \ 48'}{\sin. 90°} = \frac{a}{c};$$

or $a = c . \frac{\sin. 48° \ 48'}{r}$ and, consequently, a may be determined exactly as b has been; or thus, from b,

$$a = \sqrt{(c^2 - b^2)} = \sqrt{\{(c + b)(c - b)\}}.$$

$$\text{In Log}^{ms}. \log. a = \frac{1}{2} \{\log. (c + b) + \log. (c - b)\}.$$

3d Case, in which a side and the acute angle (which is not opposite to it) are given.

Here, b, A and $C = 90°$ are given, and a, c, B, are required.

Solution.

B determined.

$$B = 90° - A$$

a determined.

$$\frac{\sin. A}{\sin. B} = \frac{a}{b} \text{ [Cor. 1. Prob. 2.]}$$

$$\therefore a = b . \frac{\sin. A}{\sin. B} = b . \frac{\sin. A}{\cos. A}$$

$$= \frac{b \tan. A}{r};$$

$\therefore$ in logarithms,

$$\log. a = \log. b + \log. \tan. A - 10$$

c determined.

$$\frac{\sin. C}{\sin. B} = \frac{c}{b}; \therefore c = b . \frac{\sin. C}{\sin. B}$$

since $C = 90°$ and $\sin. B = \cos. A$

$$c = \frac{b}{\cos. A} = \frac{r b}{\cos. A} \text{ (rad.} = r)$$

In logarithms,

$$\log. c = 10 + \log. b - \log. \cos. A.$$

Example.

$b = 31.76$ $A = 17° \ 12' \ 51''$
$\therefore B = 72° \ 47' \ 9''$.

Computation for a.

$\log. b$ or $\log. 31.76 = 1.5018805$
$\log. \tan. 17° \ 12' \ 51'' = 9.4911132$

$\log. a + 10 = 10.9929937$
$\therefore a = 9.8399$

Computation for c.

$\log. 10^{10} = 10$
$\log. 31.76 = 1.5018805$

$10 + \log. b. = 11.5018805$
$\log. \cos. 17° \ 12' \ 51'' = 9.9800967$

$\log. c = 1.5217838$

$\therefore c = 33.249.$

We now proceed to the *Cases of Oblique-angled Triangles.*

First Case, in which two angles and a side opposite to one of the angles are given.

Here, A, B, a, are given, and b, c, C are required.

Solution.

C determined.

$A+B+C=180^\circ \therefore C=180^\circ-(A+B)$

b determined.

by Cor. 1, Prob. 2.

$$\frac{\sin. B}{\sin. A}=\frac{b}{a};\ \therefore b=a.\frac{\sin. B}{\sin. A}$$

In logarithms,

$\log. b=\log. a+\log. \sin. B-\log. \sin. A$

The side c is similarly determined.

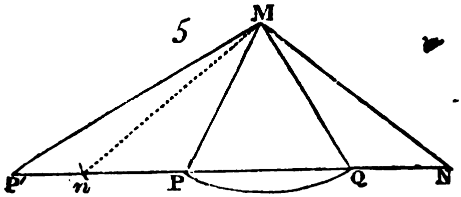
Example.

$A=41^\circ\, 13'\, 22''$, $B=71^\circ\, 19'\, 5''$, $a=55$

$\therefore C=180^\circ-(112^\circ\, 32'\, 27'')=67^\circ\, 27'\, 33''$

$$\begin{aligned}
&\log. a \text{ or } \log. 55 = 1.7403627\\
&\log. \sin. B \text{ or } \log. \sin. 71^\circ\, 19'\, 5'' = 9.9764927\\
\hline
&\log. a+\log. \sin. B = 11.7168554\\
&\log. \sin. A = 9.8188779\\
\hline
&\therefore \log. b = 1.8979775\\
&\therefore b = 79.063.
\end{aligned}$$

Second Case of oblique-angled triangles, in which, two sides and an angle opposite to one of the sides are given.



Here, a, b, B are given, and A, C, c are required.

Solution.

A determined.

$$\frac{\sin. A}{\sin. B}=\frac{a}{b}\ \therefore \sin. A=\sin. B\,\frac{a}{b}.$$

In logarithms,

$\log. \sin. A=\log. \sin. B+\log. a-\log. b$

This case may be ambiguous, or will admit of two solutions, when $a > b$, and B is acute: for, let $MN=a$, $MP=b$, $\angle MNP=B$, take $Mn=MN$, then $MP(<Mn)$ falls between N, n,

Example 1. (ambiguous).

$a=178.3$, $b=145$, $B=41^\circ\, 10'$

Computation for A.

$$\begin{aligned}
&\log. \sin. B \text{ or } \log. \sin. 41^\circ\, 10'=9.8183919\\
&\log. a \text{ or } \log. 178.3=2{,}2511513\\
\hline
&12.0695432\\
&\log. b \text{ or } \log. 145=2.1613680\\
\hline
&\log. \sin. A=9.9081752
\end{aligned}$$

$\therefore$

$N, n,$ and another line MQ, also between N, n, can be taken equal to it; $\therefore$ the triangle may be MNP, or MNQ, and the angle A may be either MPQ or its supplement MQN; but, if b or MP be $> a$ or MN, MP would fall beyond N and n, as MP' does, and no other line equal to it can be drawn between P' and N: in this case, A has one value only. If B be obtuse, A cannot; therefore here also the case is not ambiguous.

A and B being known, $C = 180^\circ - (A + B)$ is known.

$$\therefore A = 54^\circ\ 2'\ 22''$$
$$\text{and } C = 84\ 47\ 38$$
$$\text{or } A = 125\ 57\ 38$$
$$\text{and } C = 12\ 52\ 22$$

Example 2. (not ambiguous)

$a = 145$, $b = 178.3$, $B = 41^\circ\ 10'$

log. sin. $41^\circ\ 10' = 9.8183919$

log. $145 = 2.1613680$

11.9797599

log. $178.3 = 2.2511513$

log. sin. $A = 9.7286086$

$\therefore A = 32^\circ\ 21'\ 54''$.

In this instance the supplement of A cannot belong to the case proposed.

c determined.

$$\therefore \frac{\sin. C}{\sin. B} = \frac{c}{b} \therefore c = b \cdot \frac{\sin. C}{\sin. B};$$

or c may directly, that is without the intervention of the process for finding A, be determined from this expression

$$\cos. B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ whence,}$$
$$c = a \cos. B \pm \sqrt{(b^2 - a^2 \sin.^2 B)}.$$

Computation of c in 1st Example.

log. $145 = 2.1613680$

log. sin. $84^\circ\ 47'\ 38''$ 9.9982047

12.1595727

log. sin. $41^\circ\ 10' = 9.8183919$

log. $c = 2.3411808$

$\therefore c = 219.37$

Third Case, in which two sides and the included angle are given,

Here, a, b, C are given, and A, B, c are required.

By Cor. 1, Prob. 2. — $\frac{a}{b} = \frac{\sin. A}{\sin. B}$

$$\therefore \frac{a}{b} - 1 = \frac{\sin. A}{\sin. B} - 1$$
$$\text{or } \frac{a-b}{b} = \frac{\sin. A - \sin. B}{\sin. B},$$
$$\text{similarly } \frac{a+b}{b} = \frac{\sin. A + \sin. B}{\sin. B};$$

$\therefore$

Example.

$a = 562$, $b = 320$, $C = 128^\circ\ 4'$.

Computation.

$a - b = 242$

$a + b = 882$

$A + B = 180^\circ - (128^\circ\ 4') = 51^\circ\ 56'$

$\frac{A+B}{2} = 25^\circ\ 58'$.

log.

$\therefore$

$$\frac{a-b}{a+b}=\frac{\sin. A-\sin. B}{\sin. A+\sin. B}=\frac{\tan.\frac{A-B}{2}}{\tan.\frac{A+B}{2}}$$

by [*f*] page 33

compute $\therefore \tan.\frac{(A-B)}{2}$ from this expression, and $\frac{A-B}{2}$ is known from the Trigonometrical Tables, and since

$A+B=180^\circ-C$ we shall have A and B, for if

$$\frac{A+B}{2}=s, \text{ and } \frac{A-B}{2}=d$$

then, $A=s+d$, and $B=s-d$.

c determined.

$$\frac{\sin. C}{\sin. A}=\frac{c}{a} \therefore c=a.\frac{\sin. C}{\sin. A}.$$

In logarithms, $\log. c=$

$\log. a+\log. \sin. C-\log. \sin. A$,

or *c* may directly be found, thus:

$$\cos. C=\frac{a^2+b^2-c^2}{2ab}$$

$$\therefore c=\sqrt{(a^2+b^2-2ab\cos. C)}$$

which form however is not suited to logarithmic computation.

$\log. \tan. 25^\circ\ 58' = 9.6875402$
$\log. 242 \qquad = \underline{2.3838154}$
$\qquad\qquad\qquad 12.0713556$
$\log. 882 \qquad = \underline{2.9454686}$

$$\log. \tan.\frac{A-B}{2}=9.1258870$$

$$\therefore \frac{A-B}{2}=7^\circ\ 36'\ 40''$$

$$\frac{A+B}{2}=25^\circ\ 58'$$

$\therefore A=33^\circ\ 34'\ 40''$
$\quad B=18^\circ\ 21'\ 20''$

Computation of *c*.

$\log. 562 = 2.7497363$
$\left.\begin{array}{l}\log. \sin. 128^\circ\ 4' \\ \text{or } \log. \sin. 51\ \ 56\end{array}\right\} = 9.8961369$
$\qquad\qquad\qquad \overline{12.6458732}$
$\log. \sin. 33^\circ\ 34'\ 40'' = \underline{9.7427789}$
$\log. c = 2.9030943$
$\therefore c = 800.01$

The above is a complete solution of the case, in which two sides and the included angle are the quantities given. But, the analytic art is required to furnish, besides merely adequate solutions, commodious and concise ones. And, of this latter character are the solutions which have been given of the third case by Dr. Maskelyne, in the Introduction to Taylor's *Logarithms*, and by Legendre, in his *Trigonometry*, p. 369, 4th edit. These solutions we now proceed to explain.

Let $a > b$: find in the Tables an angle θ, such that tan. $\theta = r\frac{a}{b}$ and from this logarithmic expression:

$$\log.\tan.\theta = 10 + \log.a - \log.b \ldots\ldots\ldots\ldots [\alpha]$$

in this case, since $a > b$, θ is $> 45^0$, for when $\theta = 45^0$, tan. $\theta = r$.

Now, since $\tan.\theta = r.\frac{a}{b}$, $\tan.\theta \mp r = r\left(\frac{a}{b} \mp 1\right) = r\frac{a \mp b}{b}$;

$$\therefore \frac{\tan.\theta - r}{\tan.\theta + r} = \frac{a-b}{a+b} = \frac{\tan.\left(\frac{A-B}{2}\right)}{\tan.\left(\frac{A+B}{2}\right)},$$

but by Prob. 4, pp. 35, 36, &c. $\tan.(\theta - 45^0) = r\left(\frac{\tan.\theta - r}{\tan.\theta + r}\right)$

hence, $r\tan.\left(\frac{A-B}{2}\right) = \tan.\left(\frac{A+B}{2}\right).\tan.(\theta - 45^0) \ldots\ldots [\beta]$

consequently, since θ is known from the expression $[\alpha]$, $\tan.\frac{A-B}{2}$ may be computed; and thence, by means of the Trigonometrical Tables, $\frac{A-B}{2}$ is known, and A and B may be determined as in the former case.

Solution of the preceding Example by this method.

Computation of θ by the formula $[\alpha]$.

$$10 + \log.562 = 12.7497363$$
$$\log.320 = 2.5051500$$
$$\therefore \log.\tan.\theta = 10.2445863 \quad \therefore \theta = 60^\circ\ 20'\ 35''$$
$$\theta - 45^\circ = 15^\circ\ 20'\ 35''.$$

Computation of $\frac{A-B}{2}$ by the formula $[\beta]$

$$\log.\tan.15^\circ\ 20'\ 35'' = 9.4383476$$
$$\log.\tan.\frac{A+B}{2}, \text{ or } \log.\tan.25^\circ\ 58' = 9.6875402$$
$$\therefore 10 + \log.\tan.\frac{A-B}{2} = 19.1258878$$

$$\therefore \frac{A-B}{2} = 7° \ 36' \ 40''$$

$$\text{and since } \frac{A+B}{2} = 25° \ 58'$$

$$A = 33° \ 34' \ 40'' \text{ and } B = 18° \ 21' \ 20''$$

the same result as was obtained by the preceding method, and obtained in practice, almost with equal facility, even when it is necessary to take from the Tables the logarithms of a and b.

The demonstration of this method is not more concise than that of the preceding; but, the rule and the connected computation are, and, especially, in those cases in which the logarithms of a and b should happen to be given; for then θ would immediately be determined from the form [α], and $\frac{A-B}{2}$ from the form [β], so that the whole of the rule would be expressed by the two forms [α], [β].

The above method of determining the angles A, B, is the same as that which is contained in the fourth Proposition of Robert Simpson's *Trigonometry*, p. 486, of his Elements of Euclid, 6th edition: and, in substance, is the same as the method given by Dr. Maskelyne, p. 36. Introduction to Taylor's *Logarithms:* the sole difference of the two methods is in the expression: instead of the formula,

$$r \cdot \tan. \left(\frac{A-B}{2}\right) = \tan. \left(\frac{A+B}{2}\right) \tan. (\theta - 45°)$$

Dr. Maskelyne directs us to employ

$$r \cdot \tan. \left(\frac{A-B}{2}\right) = \text{co-tan.} \frac{C}{2} \cdot \tan. (\theta - 45°)$$

$$\text{but, since } A+B+C = 180, \ \frac{C}{2} = \left(90° - \frac{A+B}{2}\right)$$

$$\text{consequently, co-tan.} \frac{C}{2} = \tan. \left(\frac{A+B}{2}\right).$$

Two methods of computing the side c, have been already given, one, from A and B previously determined; the other, independent of such determination: the latter method, however, is not adapted to logarithmic computation; but, it may be by the introduction of an angle, called a subsidiary angle, such as θ is in the preceding demonstration, thus: from page 82,

$$\begin{aligned} c &= \sqrt{(a^2 + b^2 - 2\,a\,b\,.\cos.\ C)} \\ &= \sqrt{(a^2 - 2\,a\,b + b^2 + 2\,a\,b - 2\,a\,b\,.\cos.\ C)} \\ &= \sqrt{\{(a-b)^2 + 2\,a\,b\,(1 - \cos.\ C)\}} \\ &= (a-b)\sqrt{\left(1 + \frac{2\,a\,b}{(a-b)^2}\ \text{ver. sin. } C\right)} \end{aligned}$$

Assume $\dfrac{4ab}{(a-b)^2}\,\dfrac{\text{ver. sin. } C}{2} = \tan.^2\theta$, in which case

$$1 + \frac{4\,a\,b}{(a-b)^2} \cdot \frac{\text{ver. sin. } C}{2} = 1 + \tan.^2\theta = \sec.^2\theta\,;$$

consequently,

$$c = (a - b)\,.\sec.\ \theta, \text{ or } = \frac{a - b}{\cos.\ \theta}\ (\text{rad.} = 1)$$

$$\text{or } c = (a-b)\,\frac{\sec.\ \theta}{r}, \text{ or } = \frac{(a - b)\,r}{\cos.\ \theta}\ (\text{rad.} = r)$$

and in logarithms,

$$\log.c = \log.(a-b) + \log.\sec.\theta - 10, \text{ or } = \log.(a-b) + 10 - \log.\cos.\theta.$$

This agrees with Dr. Maskelyne's determination of c, given in page 36 of his Introduction to Taylor's *Logarithms*: and the sole difference in the process is that, instead of

$$\frac{\text{ver. sin. } C}{2}, \text{ he uses, } \sin.^2\frac{C}{2};$$

which two values, as it appears from p. 43, are equal.

Example of the computation of c: a and b being 562 and 320, and $C = 128^\circ\ 4$.

$$2\log.\tan.\theta - 10 = \log.\ 2 + \log.a + \log.b + \log.\text{ ver. sin. } C - 2\log.(a-b)$$

$$\begin{aligned} \text{Now, } \log.\ 2 &= 0.3010300 \\ \log.\ 562 &= 2.7497363 \\ \log.\ 320 &= 2.5051500 \\ \log.\text{ ver. sin. } 128^\circ\ 4' &= 10.2085966 \\ \hline & \ 15.7645129 \\ 2\log.\ 242 &= 4.7676308 \\ \hline & \ 10.9968821 \end{aligned}$$

$$\therefore \text{log. tan. } \theta = 10.4984410; \therefore \text{log. sec. } \theta = 10.5192823$$

$$\therefore \text{log. } c = 10.5192823 + 2.3838154 - 10 = 2.9030977$$

$$\therefore c = 800.01.$$

If we make the value of c equal to

$$\sqrt{(a^2 + 2ab + b^2 - 2ab - 2ab \cos. C)}$$

we shall have $c = (a+b)\sqrt{\left(1 - \frac{2ab}{(a+b)^2}(1 + \cos. C)\right)}$

and, if we assume $\frac{4ab}{(a+b)^2} \cdot \left(\frac{1 + \cos. C}{2}\right) = \sin.^2 \theta$,

there will result, $c = (a+b).\cos. \theta$,

$1 + \cos. C$ is the versed sine of the supplement of C, which Mr. Mendoza, in his valuable Tables on *Nautical Astronomy*, calls the * *suversed sine;* hence the rule for the solution algebraically expressed is, log. sin. θ

$$= \tfrac{1}{2}\{10 + \text{log. } 2 + \text{log. suver. sin. } C + \text{log. } a + \text{log. } b - 2 \text{ log. } (a+b)\}$$

$$\text{log. } c = \text{log. } (a+b) + \text{log. cos. } \theta - 10.$$

Of the preceding solutions of the third Case, one alone, as it has been remarked, is, in strictness, sufficient: the others have been added, for the sake of rendering, in certain cases, the computation more expeditious. And, when a specific instance is presented, it will not be difficult to determine which method of solution it is, that ought to be adopted. If, for example, the side opposite the included angle be alone required, we ought to compute it by the method of p. 85, l. 14, avoiding, as unnecessary, the calculation of the angles (A, B,) at the base.

Fourth Case, in which the three sides are given.

Here, a, b, c, are given, and A, B, C, are required.

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* In these Tables the log. suversed sine = log. $\frac{1 + \cos.}{2}$.

First Solution.

By Prob. 2, page 26, sin. A =

$$\frac{2}{bc}\sqrt{\left\{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}\right)-c\right\}}$$

and, in logarithms, putting $\frac{a+b+c}{2} = S$, and instead of sin. A, $\frac{\text{sin. } A}{r}$,

$$\text{log. sin. } A - 10 = \text{log. } 2 + \tfrac{1}{2}\left\{\text{log. } S + \text{log.}(S-a) + \text{log.}(S-b) + \text{log.}(S-c)\right\} - \text{log. } b - \text{log. } c$$

and similarly, log. sin. C is expressed by the same form, substituting in the negative part, instead of − log. c, − log. a, so is also

log. sin. B, substituting instead of −log. b, −log. a.

Second Solution.

By Prob. 2, page 26.

$$1 - \text{cos. } A = \frac{(a+b-c)(a+c-b)}{2bc}$$

$$= \frac{4}{2bc}\left\{\left(\frac{a+b+c}{2}-c\right)\left(\frac{a+b+c}{2}-b\right)\right\}$$

But $1 - \text{cos. } A = 2 . \text{sin.}^2 \frac{A}{2}$, [page 42.] hence, introducing the radius r,

$$\frac{\text{sin.}^2 . \frac{A}{2}}{r^2} = \frac{(S-b)(S-c)}{bc}.$$

In logarithms,

$$2 \text{ log. sin. } \frac{A}{2} = 20 + \text{log.}(S-b) + \text{log.}(S-c) - \text{log. } b - \text{log. } c,$$

and similarly,

$$2 \text{ log. sin. } \frac{C}{2} = 20 + \text{log.}(S-a) + \text{log.}(S-b) - \text{log.}\, a - \text{log.}\, b.$$

Third Solution.

By Prob. 2, page 26,

$$1 + \text{cos. } A = \frac{(a+b+c)(c+b-a)}{2bc} = 2\frac{S.(S-a)}{bc}.$$

But $1 + \text{cos. } A = 1 - \text{cos. } (\pi - A) = 2 \text{ sin.}^2 \frac{\pi - A}{2} = 2 \text{ sin.}^2 \left(\frac{\pi}{2} - \frac{A}{2}\right)$

$= 2 \,.\, \text{cos.}^2 \frac{A}{2}$. Hence,

$$\text{cos.}^2 . \frac{A}{2} = r^2 \times \frac{S.(S-a)}{bc}.$$

In logarithms,

$$2 \text{ log. cos. } \frac{A}{2} = 20 + \text{log. } S + \text{log. } (S-a) - \text{log. } b - \text{log. } c$$

and similarly,

$$2 \text{ log. cos. } \frac{C}{2} = 20 + \text{log. } S + \text{log. } (S-c) - \text{log. } a - \text{log. } b.$$

Fourth Solution.

Divide the expression for the $\text{sin.}^2 \frac{A}{2}$ by that for $\text{cos.}^2 \frac{A}{2}$, and since the tangent is equal to the radius multiplied into a fraction of which the numerator is the sine and the denominator the cosine, we have, $\text{tan.}^2 \frac{A}{2} = r^2 \times \frac{(S-b)(S-c)}{S.(S-a)}$.

In logarithms,

$$2 \text{ log. tan. } \frac{A}{2} = 20 + \text{log. } (S-b) + \text{log. } (S-c) - \text{log. } S - \text{log. } (S-a)$$

and similarly,

$$2 \text{ log. tan. } \frac{C}{2} = 20 + \text{log. } (S-b) + \text{log. } (S-a) - \text{log. } S - \text{log. } (S-c).$$

Example by the first method of solution.

$$\left.\begin{array}{l} a=33 \quad \log.=1.5185139 \\ b=42.6\ \log.=1.6294096 \\ \underline{c=53.6}\ \log.=1.7291648 \end{array}\right\} \begin{array}{ll} \therefore \log. a+\log. b=3.5479235 & [C]* \\ \log. b+\log. c=3.3585744 & [A] \\ \log. a+\log. c=3.2476787 & [B] \end{array}$$

$$\begin{array}{ll} \frac{a+b+c}{2} \quad =64.6 & \log.=1.810325 \\ \frac{a+b+c}{2}-a=31.6 & \log.=1.4996871 \\ \frac{a+b+c}{2}-b=22 & \log.=1.3424227 \\ \frac{a+b+c}{2}-c=11 & \log.=1.0413927 \\ \hline & 2 \mid 5.6937350 \mid \\ & 2.8468675 \\ (10+\log. 2)\ldots\ldots & 10.3010300 \\ \hline & 13.1478975. \end{array} \left\{\begin{array}{l} =\log.\text{ area: Cor. 2. Prob. 2.} \\ \therefore \text{ area} = 702.858 \end{array}\right.$$

Hence, (see the formula of solution) if we subtract from this logarithm (13.1478975) the values [*C*], [*A*], [*B*], we shall have, respectively, the log. sines of the angles *C*, *A* and *B*:

$$\begin{array}{lll} \ 13.1478975 & \ 13.1478975 & \ 13.1478975 \\ [C]\ \ \underline{3.1479235} & [A]\ \underline{3.3585744} & [B]\ \underline{3.2476787} \\ 9.9999740=\log. \sin. C & 9.7893231=\log. \sin. A & 9.9002188=\log. \sin. B \end{array}$$

$$\begin{array}{lll} \therefore C=89^\circ\ 22'\ 21'' & A=37^\circ\ 59'\ 53'' & B=52^\circ\ 37'\ 46''\tfrac{9}{16} \\ \text{or } 89\ \ 22\ \ 22 & B=52\ \ 37\ \ 46\tfrac{9}{16} & \\ \text{to } 89\ \ 22\ \ 25 & \therefore \underline{C=89\ \ 22\ \ 20\tfrac{7}{16}} & \\ & \underline{\quad 180\ \ \ 0\ \ \ 0\quad} & \end{array}$$

In this case, if we had not determined *A* and *B*, the value of *C* (determined from its logarithm of seven places of figures) would have been doubtful to the extent of 4″: consequently, for the finding the exact value of *C*, in this instance, the first method is not proper.

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* [*C*] [*A*] [*B*] are merely marks of reference.

By the second method; the same Example.

Angle C computed.

$$\begin{array}{lr} \log.(S-a) = & 1.4996871 \\ \log.(S-b) = & 1.3424227 \\ \hline \text{(20 added)} & 22.8421098 \\ [C] \text{ or } \log. a + \log. b = & 3.1479235 \\ \hline \end{array}$$

$$2 \log. \sin. \frac{C}{2} = \quad 19.6941863$$

$$\log. \sin. \frac{C}{2} = \quad 9.8470931$$

$$\therefore C = 2\,(44^{\circ}\ 41'\ 10''\tfrac{4}{21})$$
$$= \quad 89 \quad 22 \quad 20\tfrac{8}{21}$$

Angle A computed.

$$\begin{array}{lr} \log.(S-b) = & 1.3424227 \\ \log.(S-c) = & 1.0413927 \\ \hline \text{(20 added)} & 22.3838154 \\ [A] & 3.3585744 \\ \hline \end{array}$$

$$2 \log. \sin. \frac{A}{2} = 19.0252410$$

$$\log. \sin. \frac{A}{2} = \quad 9.5126205$$

$$\therefore A = 2\,(18^{\circ}\ 59'\ 56''\tfrac{30}{61})$$
$$= \quad 37 \quad 59 \quad 52\tfrac{60}{61}.$$

By this method the angle C is determined at once with great accuracy.

By the third method.

Angle C computed.

$$\begin{array}{lr} \log. S = & 1.8102325 \\ \log.(S-c) = & 1.0413927 \\ \hline \text{(20 added)} & 22.8516252 \\ [C] & 3.1479235 \\ \hline & 19.7037017 \end{array}$$

$$\log. \cos. \frac{C}{2} = 9.8518508{,}5$$

$$\therefore C = 2(44^{\circ}\ 41'\ 10\tfrac{45}{210})$$
$$= \quad 89 \quad 22 \quad 20\tfrac{9}{21}$$

Angle A computed.

$$\begin{array}{lr} \log. S = & 1.8102325 \\ \log.(S-a) = & 1.4996871 \\ \hline \text{(20 added)} & 23.3099196 \\ [A] & 3.3585744 \\ \hline & 19.9513452 \end{array}$$

$$\log. \cos. \frac{A}{2} = 9.9756726$$

$$\therefore A = 2\,(18^{0}\ 59'\ 56''.5)$$
$$= \quad 37 \quad 59 \quad 53.$$

As C, in these two last methods, is determined to a great degree of exactness, the value of B is not computed from the formula, but it may be had by subtracting $A+C$ from 180°.

By the fourth method.

Angle C computed.

$$\begin{aligned} \log.\,(S-b) &= 1.3424227 \\ \log.\,(S-a) &= \underline{1.4996871} \\ (20 \text{ added})\ & 22.8421098\ (c) \end{aligned}$$

Again,

$$\begin{aligned} \log.\,S &= 1.8102325 \\ \log.\,(S-c) &= \underline{1.0413927} \\ & 2.8516252\ (d) \end{aligned}$$

$$\therefore (c)-(d) = 19.9904846$$

$$\therefore \log.\,\tan.\frac{C}{2} = 9.9952423$$

$$\therefore C = 2(44^\circ\ 41'\ 10''\tfrac{9}{41})$$

$$= 89\ \ 22\ \ 20\tfrac{18}{41}$$

Angle A computed.

$$\begin{aligned} \log.\,(S-b) &= 1.3424227 \\ \log.\,(S-c) &= \underline{1.0413927} \\ (20 \text{ added})\ & 22.3838154\ (a) \end{aligned}$$

Again,

$$\begin{aligned} \log.\,S &= 1.8102325 \\ \log.\,(S-a) &= \underline{1.4996871} \\ & 3.3099196\ (b) \end{aligned}$$

$$\therefore (a)-(b) = 19.0738958$$

$$\therefore \log.\,\tan.\frac{A}{2} = 9.5369479$$

$$\therefore A = 2(18^\circ\ 59'\ 56''.5)$$

$$= 27\ \ 59\ \ 53.$$

As far as instances prove, any one of these three latter methods may be used for determining the angle C, and angles nearly of the same magnitude; and, it is of no material consequence which it is that is used. The first method is plainly, from a mere comparison of results, insufficient to give exactly the value of such an angle as C is: and we need not go through the labour of the arithmetical computation in order to ascertain its insufficiency: for, if we perceive that the square of the side, such as c, is nearly equal to the sum of the squares of the other two sides, we shall know that the value of C does not differ much from 90°. *

It may now be worth the while to enquire, more minutely, why, since compendium of calculation is a desirable object, several methods of solution have been given.

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* This 4th case of oblique triangles is commonly (see Robert Simpson's *Euclid*, page 488; Ludlam, page 220,) solved by means of this proposition. The sum of the two sides of a triangle is to the base as the difference of the segments of the base is to the difference of the sides; but the demonstration of this proportion, since the case is otherwise more conveniently solved, is purposely omitted.

Now, each of the preceding methods is adapted to logarithmic computation, and each, in an analytical point of view, affords a complete solution. One solution, therefore, would have been sufficient, and one alone given, if the same applied, with equal convenience and equal numerical accuracy, to all instances; but the fact is otherwise. If an example were proposed in which the angle A should be nearly 90°, as C is in the former Example: the log. sin. A might be deduced from the first solution; but, to such logarithmic value, there would not, in the Tables, correspond a precise value of the angle A: for instance, if the numerical value of log. sin. A should be 9.9999998, A might equal (by the Tables) either 89° 56′ 19″, or 89° 57′ 8″, or any angle intermediate of these two angles. The reason of this is, the very small variation of the sine of an angle nearly equal to 90°. And, this small variation is apparent from the mere inspection of the Geometrical diagram, in which two contiguous sines should be drawn to two arcs each nearly equal to a quadrant; or, analytically, it may be thus shewn. Let A be an arc nearly $=90°$; let it be increased by a small quantity (1″ for instance), then by the formula [1], p. 27, making $B = 1''$,

$$\sin.(A+1'') = \sin. A . \cos. 1'' + \cos. A . \sin. 1''.$$

Subtract sin. A from each side of the equation, then

$$\sin.(A+1'') - \sin. A = \sin. A\,(\cos. 1'' - 1) + \cos. A \sin. 1'',$$

$$\text{(by p. 42,)} \qquad = -\sin. A . 2 \sin.^2 \frac{1''}{2} + \cos. A \sin. 1''.$$

Now, $\sin. A \times 2 \sin.^2 \frac{1''}{2}$ may, from the smallness of the factor $2 \sin.^2 \frac{1''}{2}$, be neglected; and, accordingly,

$$\sin.(A + 1'') - \sin. A = \cos. A . \sin. 1'', \text{ nearly};$$

therefore, the difference of two contiguous sines, or what has been called, the variation of the sine, varies nearly as cos. A; and the cos. A is when $A = 90°$, nearly, a very small quantity relatively to its other values, in which A is of a mean value.

It must not, however, be unnoticed, that the want of precision in the determination of the angle is partly owing to the construction of the Logarithmic and Trigonometrical Tables. The Tables referred to, and in common use *, are computed to seven places of figures; but, if we had Tables † computed to a greater number of places, to double the number, for instance, then the logarithmic sines of all angles between 89° 56′ 18″, and 89° 57′ 9″, would not be expressed, as they are in Tables now in use, by the same figures. In such circumstances, we should obtain conclusions very little remote from the truth; but then, such Tables would be extremely incommodious for use, and would, in all common cases, give results to a degree of accuracy quite superfluous and useless. Moreover, such Tables, even in the extreme cases which we have mentioned, are not essentially necessary: since their use can be superseded, by abandoning the first method of solution, and recurring either to the 2d, 3d, or 4th method.

When the angle (A) sought then is nearly $= 90°$, the first method must not be used, but one of the latter methods, in which either the sine, cosine or tangent of *half* the angle is determined; and, in such an extreme case, it is a matter of indifference whether, instead of the first method, we substitute the 2d, or 3d, or 4th. But, in other cases, it is not a matter of indifference: for since, as it has been shewn, the variation or the increment of the sine is as the cosine, and of the cosine as the sine, these two variations are equal at 45°, but beyond 45°, up to 90°, that of the sine is less, and that of the cosine greater;

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* Shirwin's 8vo. Hutton's 8vo. Taylor's 4to.

† In Vlacq's Tables, published at Gouda, 1633, we have

Arcs.	Log. Sines.
89° 56′ 10″	9.9999997300
20	97530
40	97958
50	98154
89 57 0	98346
10	98525

and, the contrary happens between 45° and 0; consequently we have this Rule:

If the angle sought be $< 90^{\circ}$, use the second method;
if .... $> 90^{\circ}$, use the third method.

The 4th method may be used, and commodiously, for all values of the angles sought from 0 up to angles nearly $= 180^{\circ}$: when, however, the angle (A) is nearly $= 180^{\circ}$, $\tan. \frac{A}{2}$, which is nearly $\tan. 90^{\circ}$, is very large, and its variations, (which are as the square of the secant *) are also very large and irregular. If, therefore, we use Sherwin's Tables, which are computed for every minute only of the quadrant, the logarithms corresponding to the seconds, taken out by proportional parts, will not be exact: for, in working by proportional parts, it is supposed, if the difference between the logarithmic tangents of two arcs differing by 60 seconds be d, that the difference between the logarithmic tangents of the first arc, and of another arc, that differs from it only by n seconds is $\frac{n}{60} d$: now, this is not true for arcs nearly equal to 90°; and an example will most simply shew it: by Sherwin's Tables,

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* For by the formula, p. 35,

$$\tan. (A + 1'') = \frac{\tan. A + \tan. 1''}{1 - \tan. A. \tan 1''}.$$

Subtract tan. A from each side, and

$$\tan. (A + 1'') - \tan. A = \frac{\tan. 1'' + \tan.^2 A \tan. 1''}{1 - \tan. A . \tan. 1''}$$

$$\left.\begin{array}{l}\text{by expanding and neglect-}\\ \text{ing terms involving } \tan.^2 1''\end{array}\right\} = (\tan. 1'' + \tan.^2 A. \tan. 1'')(1 + \tan. A \tan. 1'')$$

$$= \tan. 1'' (1 + \tan.^2 A) \text{ nearly};$$

$\therefore$ since tan. 1″ is an assigned quantity,

$$\tan. (A + 1'') - \tan. A \propto 1 + \tan.^2 A \propto \sec.^2 A.$$

log. tan. 89° 30′ = 12.0591416
log. tan. 89 29 = 12.0449004................[2]

log. diff. corresponding to 60″ = 142412
diff. corresponding to 30″ = 71206 ....[4]
∴ by Rule, log. tan. 89° 29′ 30″ ([2] + [4]) = 12.0520210
whereas true log. tan. 89° 29′ 30′, by Taylor's Log. = 12.0519626

Again,

log. tan. 89° 50′ = 12.5362727
log. tan. 89 49 = 12.4948797

log. diff. corr^{g}. to 60″ = 413930
∴ diff. corr^{g}. to 6″ = 41393
∴ by the Rule log. tan. 89° 49′ 6″ = 12.4990190
whereas the true log. tan. 89° 49′ 6″, by Taylor's Log^{ms}. = 12.498845

In these instances, the log. tangent, determined by the proportional parts, is too large, which it plainly must be; for, the logarithmic increment of the tangent increasing as the arc does, that is, the increment during the last 30″ being greater than the increment during the first 30″, if we take half the whole increment for the increment due to the first 30″, or one-tenth of the whole increment, for the increment due to the first 6″, we plainly take quantities too large. The same reason would, it is true, hold against calculating logarithmic tangents of any arcs by proportional parts, if the values of logarithmic tangents were exactly put down in Tables; but (we speak of the Tables in ordinary use) the values are expressed by seven places only of figures; and, as far as seven places, the irregularities in the successive differences of the logarithmic tangents of arcs that are of some mean value, between 0 and 90°, do not appear; thus, by Sherwin's Tables,

log. tan. 44° 30′ = 9.9924197
log. tan. 44 29 = 9.9921670

log. diff. corresponding to 60″ = 2527
∴ diff. corresponding to 30 = 12635
∴ by the Rule log. tan. 44° 29′ 30″ = 9.99229335
and the true log. tan. by Taylor's Tables = 9.9922934.

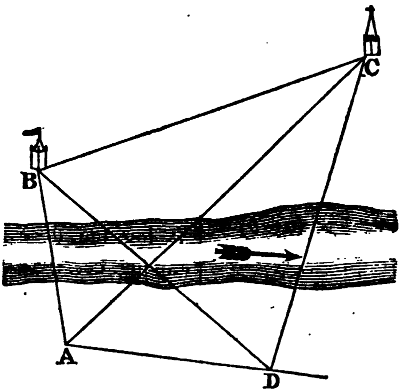
It appears then, from the assigned reason, and by the instances given, that an angle nearly 90° cannot exactly be found from its logarithmic tangent. The determination of the angle by means of proportional parts will be wrong in seconds by Sherwin's Tables; and will be wrong in the parts of seconds by Taylor's Tables. From the whole of what has been said then, it appears that in computing the values of angles, two inconveniences may occur, either when the successive logarithmic numbers are too nearly alike, as in the case of sines of angles nearly 90°, or too widely different, as in the case of the tangents of angles nearly equal to 90°. It is the business of the Analyst to provide formulæ, by which these inconveniences may be remedied or avoided; and hence have arisen the different methods for attaining, apparently, the same end.

Before we entirely relinquish this digression, we wish to observe, that, although the log. sine or log. tangent of the angle *A* may be determined exactly either by the first or the fourth method, yet, if it should be very small, its value cannot, with sufficient exactness, be determined by the Tables in common use. For, very small angles cannot be exactly found from their logarithmic sines and tangents; not exactly in seconds, by Sherwin's Tables, nor exactly, in parts of seconds, by Taylor's Tables; and therefore, as great exactness may be required, and is commonly required, in those cases, in which a very small angle is to be determined, the Tables are not to be used. They are to be superseded by a peculiar computation, of which, without demonstration, Dr. Maskelyne has given the rule in his Introduction to Taylor's *Logarithms*, p. 17 and 22. This rule and similar rules will be stated and demonstrated in a subsequent part of this Work, when the analytical series for the sine and tangent of an arc are deduced.

To the several cases of the solution of oblique triangles, examples have been given, but, merely arithmetical examples; it may proper therefore, to subjoin a feigned case of practice and observation, in order to shew, more plainly, the use and application of the formulæ of solution.

An observer at *A* wishes to determine his distance from two inaccessible objects *B*, *C*, and also the distance *BC*, of the same objects.

The observer takes a new station D, and measures the distance



AD; suppose it to equal 1763 yards: at A and D, by means of proper instruments, he makes the following observations:

$$\text{at } A, \left(\begin{matrix} BAC = 45^0\ 1'\ 3'' \\ CAD = 30^0\ 0'\ 2'' \end{matrix}\right), \text{at } D, \left(\begin{matrix} BDC = 36^0\ 15'\ 5'' \\ BDA = 33^0\ 7'\ 40'' \end{matrix}\right)$$

consequently,

$$CDA = 69^0\ 22'\ 45'',\ ACD = 180^0 - (ADC + CAD) = 80^0\ 37'\ 13''$$
$$BAD = 75^0\ 1'\ 5'',\ ABD = 180^0 - (BAD + BDA) = 71^0\ 51'\ 15''$$

AB determined.

By 1st Case of oblique triangles, p. 80.

$$\frac{AB}{AD} = \frac{\text{sin. } ADB}{\text{sin. } ABD};$$

$\therefore$ log. AB = log. AD + log. sin. ADB − log. sin. ABD.

$$\begin{array}{r} \text{log. } 1768 = 3.2462523 \\ \text{log. sin. } 33^\circ\ 7'\ 40'' = 9.7375966 \\ \hline 12.9838489 \\ \text{log. sin. } 71^\circ\ 51'\ 15'' = 9.9778456 \\ \hline \text{log. } AB = 3.0060033 \\ \therefore\ AB = 1014. \end{array}$$

AC determined.

By 1st Case of oblique triangles,

$$\frac{AC}{AD} = \frac{\text{sin. } ADC}{\text{sin. } ACD};$$

$\therefore$ log. AC = log. AD + log. sin. ADC − log. sin. ACD.

$$\begin{array}{r} \text{log. } 1763 = 3.2462523 \\ \text{log. sin. } 69^\circ\ 22'\ 45'' = 9.9712441 \\ \hline 13.2174964 \\ \text{log. sin. } 80^\circ\ 37'\ 13'' = 9.9941543 \\ \hline \text{log. } AC = 3.2233421 \\ \therefore\ AC = 1672.4. \end{array}$$

CB determined.

By the formula of computation given in page 85,

$$\frac{\tan.^2 \theta}{r} = \frac{2ab}{(a-b)^2} \text{ ver. sin. } C$$

and, log. tan. θ

$$= \frac{1}{2} \left\{ \begin{array}{l} 10 + \log. 2 + \log. a + \log. b \\ + \log. \text{versin. } C - 2 \log.(a-b) \end{array} \right\}$$

and *CB* or $c = \frac{r}{\cos. \theta} \times (a-b)$

and

$\log. c = 10 + \log. (a-b) - \log. \cos. \theta$

here, $a = AC = 1672.4$, $b = AB = 1014$

$C = BAC = 45° \, 1' \, 3''$, $a - b = 658.4$

$10 + \log. 2 =$ 10.3010300

$\log. 1672.4 =$ 3.2233421

$\log. 1014 =$ 3.0060033

$\log. \text{versin. } 45° \, 1' \, 3'' =$ 9.4670294

25.9974048

2 log. 658.4 5.6369796

20.3604252

$\therefore \log. \tan. \theta =$ 10.1801466

and $\log. \cos. \theta =$ 9.7412271 [a]

$\therefore$ since

$10 + \log. 658.4 =$ 12.8184898 [b]

$\log. c = 3.0772627 \,. [b] - [a]$

and $c = 1194.7$.

The last part of this Example (the determination of *CB*) belongs to the third case of oblique-angled triangles, in which two sides and the included angle are given; but, since the angles at the base, that is, the angles *ABC*, *ACB*, are not required, the method of solution given (see p. 82,) was not adopted: *BC*, indeed, may be computed, and, according to the common practice of calculators, would be computed, by first determining the angles *ABC*, *ACB* from the form

$$\tan. \left(\frac{ABC - ACB}{2} \right) = \frac{AC - AB}{AC + AB} \,.\, \tan. \left(\frac{ABC + ACB}{2} \right)$$

and then *BC* from this expression, $BC = AB \,.\, \frac{\sin. BAC}{\sin. ACB}$: but it is plain, that the computation, without being at all more exact, would be longer than that by which *BC* has been already determined.

The last part of the Example will also serve to illustrate the use of that solution of the third case, which was given in page 83: for, the logarithms of *AC* and *AB* being determined in the previous part, we have immediately

$\log. \tan. \theta = 10 + (3.2233421 - 3.0060033) = 10.2173388$;

whence $\theta = 58^0 \, 46' \, 23''$, and $\theta - 45^0 = 13^0 \, 46' \, 28''$;

consequently, since $\log. \tan. 13^0 \, 46' \, 23''.8 = 9.3893876$,

$$\text{log. tan.}\left(\frac{ABC+ACB}{2}\right)\text{ or co-tan. } 22^\circ\ 30'\ 31''.5 = 10.3835881\,;$$

$$\therefore \text{ log. tan. } \frac{ABC-ACB}{2} = 9.7729757$$

$$\text{and } \frac{ABC-ACB}{2} = 30^\circ\ 39'\ 48''$$

$$\text{but } \frac{ABC+ACB}{2} = 67^\circ\ 29'\ 28''.5$$

$$\text{consequently, } ABC = \overline{98^\circ\ \ 9'\ \ 16''.5}$$

$$ACB = 36^\circ\ 49'\ 40''.5.$$

The process had been somewhat more tedious if we had found these angles by the formula given in page 82, in the solution of the third case, for then we must have computed $AC - AB$, $AC + AB$, and have taken out their logarithms. This instance, and the remark on it, have been introduced to shew, not that the common and general solution is insufficient, but that other solutions may conveniently, that is, with some gain of expedition, be introduced. The Student, however, who shall peruse this Treatise in order to be initiated into Trigonometry, is advised, in the first perusal, to attend solely to the general solutions, and to postpone to a time of leisure and of acquired knowledge, the consideration of the methods that are either more expeditious, or are adapted to particular exigencies *.

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* With the view of rendering every thing as easy as possible to the Student, separate investigations of the cases of right-angled triangles, have been made to precede those of oblique-angled triangles; but, considered generally, the former cases are really included in the latter, and their solutions comprehended within the general solutions; we will shew this in two instances: suppose in the third case of oblique triangles, that the included angle is a right angle, then by the solution,

$$\text{tan.}\left(\frac{A-B}{2}\right) = \text{tan.}\left(\frac{A+B}{2}\right)\cdot\frac{a-b}{a+b};\text{ but } A+B+C = 180^\circ \text{ and } C = 90^\circ$$

$$\therefore \frac{A+B}{2} = 45^\circ;$$

consequently,

We will terminate this Chapter with one or two Problems, the solutions of which may be deduced, almost immediately, from the preceding formulæ.

1. In a triangle of which the sides are a, b, c, and the angles opposite, A, B, C, it is required to determine C, when a, b, and $A - B$ are given.

By the formula of p. 82,

$$\tan.\left(\frac{A-B}{2}\right) = \tan.\left(\frac{A+B}{2}\right)\times\frac{a-b}{a+b} = \tan.\left(90^0-\frac{C}{2}\right)\times\frac{a-b}{a+b}$$

$$= \text{(see p. 9,) } \cot.\frac{C}{2}\times\frac{a-b}{a+b} = \text{(see p. 11,)}\frac{1}{\tan.\frac{C}{2}}\times\frac{a-b}{a+b};$$

---

consequently, $\tan.\frac{A-B}{2} = \tan.45^\circ.\frac{a-b}{a+b} = \frac{a-b}{a+b}$, since $\tan.45^\circ = 1$;
but $A = 90^\circ - B$;

$\therefore \frac{A-B}{2} = 45^\circ - B$; $\therefore \tan.\left(\frac{A-B}{2}\right) = \tan.(45^\circ - B) =$ (by Prob. 4, Cor. 2, p. 35.) $\frac{1-\tan.B}{1+\tan.B}$;

hence $\frac{a-b}{a+b} = \frac{1-\tan.B}{1+\tan.B}$ and $\tan.B = \frac{b}{a}$ the same solution, in fact, as was given, page 78, in the solution of the first case of right-angled triangles.

Suppose next, in the same instance, we employ the form used in the fourth case of oblique-angled triangles, thus:

$1 + \cos.A = \dfrac{\left(\frac{a+b+c}{2}\times\frac{b+c-a}{2}\right)}{bc}$, but $\frac{b+c+a}{2}\times\frac{b+c-a}{2}$

$= \frac{b^2+c^2-a^2+2bc}{4} = \frac{b^2+bc}{2}$; since $c^2 = a^2 + b^2$:

$$\text{hence, } 1 + \cos.A = \frac{b^2+bc}{bc} = \frac{b}{c} + 1 \text{ and } \cos.A = \frac{b}{c},$$

and $\therefore c = \frac{b}{\cos.A}$ the same result as that which was obtained, page 78: so that it is plain the variety of cases might have been diminished, but not without a considerable loss of simplicity and facility.

$$\therefore \text{tan.} \frac{C}{2} = \frac{a-b}{a+b} \times \text{cot.} \frac{A-B}{2} \ldots$$

2. Given, A, B, C, and $a \pm b$; required a and b.

If the upper sign be used, then $a-b=(a+b)\times\text{tan.}\frac{C}{2}\times\text{tan.}\frac{A-B}{2}$

If the lower sign be used, then, $a+b=(a-b)\,.\,\text{cot.}\frac{C}{2}\,\text{cot.}\frac{A-B}{2}$

and in each case, by adding and subtracting, we obtain $2a$, and $2b$, and thence a and b.

3. Given A, a and $b+c$; required b, c, &c.

By the third formula of solution, p. 88,

$$2^2\,.\,\text{cos.}^2\frac{A}{2} = \frac{(a+b+c)(b+c-a)}{bc}.$$

Hence bc is known, let it $=p$, and let $b+c=s$;

$$\text{then } b+c=s$$
$$bc=p;$$

∴ (by the solution of a quadratic equation)

$$b=\frac{s}{2}+\sqrt{\left(\frac{s^2}{4}-p\right)}$$
$$c=\frac{s}{2}-\sqrt{\left(\frac{s^2}{4}-p\right)}.$$

4. Given $a+b+c$, the sum of the sides, the area of the triangle, and the angle A; it is required to find the side a (see Newton, *Arith. Univ.* Prob. 8.)

By Prob. 2, p. 26, l. 2,

$$1+\text{cos.}\,A=\frac{(b+c)^2-a^2}{2bc}=\frac{(a+b+c-a)^2-a^2}{2bc}.$$

Again, the area $(K)=\text{sin.}\,A\times\frac{bc}{2}$;

$$\therefore\ 1+\text{cos.}\,A=\frac{(a+b+c)^2-2a(a+b+c)}{4K}\times\text{sin.}\,A;$$

$$\therefore\ a=\tfrac{1}{2}(a+b+c)-\frac{2K}{a+b+c}\times\frac{1+\text{cos.}\,A}{\text{sin.}\,A}$$

$$= \tfrac{1}{2}(a+b+c) - \frac{2K}{a+b+c} \times \text{cot.}\,\frac{A}{2};$$

since (Cor. 5. p. 33.) $\dfrac{\text{sin.}\,A}{1+\text{cos.}\,A} = \text{tan.}\,\dfrac{A}{2} = \dfrac{1}{\text{cot.}\,\dfrac{A}{2}}$.

The formulæ of Trigonometry have now been applied to the resolution of rectilinear triangles; the original object, for which the science was invented. And it is to be observed, such application is the most easy, and is of very extensive practical utility. In the next Chapter we will continue to apply, still farther, the preceding formulæ. The first instances will shew their utility in expediting Arithmetical computation: the latter, selected from Works and Writings on *Physical Astronomy*, will shew their utility in subjects of great importance and of arduous investigation. In this last application, the original object of the science, the Properties of Triangles, seems entirely to be lost sight of, and the Trigonometrical analysis is peculiarly and almost solely useful, because it confers precision and power on mathematical language.

CHAP. VI.

Instances of the Utility of Trigonometrical Formulæ.

1. It is required to compute the logarithm of $a \pm b$,

$$a \pm b = a\left(1 \pm \frac{b}{a}\right).$$

Assume, for the upper sign, $\frac{b}{a} = \tan.^2\theta$, for the lower, $\frac{b}{a} = \sin.^2\theta$

then $\log.(a + b) =$

$$\log. a + \log.(1 + \tan.^2\theta) = \log. a + 2\log.\sec.\theta - 20,$$

and $\log.(a - b) =$

$$\log. a + \log.(1 - \sin.^2\theta) = \log. a + 2\log.\cos.\theta - 20.$$

In the application of this method, a and b are not supposed to be numbers: for, then, the simple way would be to add them and to take the logarithm of the sum; but a and b are compounded quantities, formed of the sines and cosines of angles: thus, in finding the Moon's distance from a star,

$$\text{versin. dist.} = \text{versin.}(d' - d) + \cos. d.\cos. d'.\text{versin.} A$$

($d.d'$ being the declinations, and A the difference of the right ascensions) a is $=$ versin. $(d' - d)$ and $b = \cos. d.\cos. d'$ versin. A, and $\tan.^2\theta$ is assumed $=$

$$\frac{\cos. d.\cos. d'\,\text{versin.} A}{\text{verin.}(d' - d)}, \text{ or, } = \frac{\cos. d.\cos. d'.\sin.^2\frac{A}{2}}{\sin.^2\frac{d' - d}{2}},$$

and then, by the process above stated, the form is adapted to logarithmic computation.

M. Sejour solves this instance somewhat differently (see his *Traité Analyt.* tom. I. p. 103.)

Thus, let $a + b = y$,

then, $a \sin. 45^0 + b \cos. 45^0 = y . \sin. 45^0$ (1).

Assume $\tan. z = \frac{b}{a}$; $\therefore \frac{\sin. z}{\cos. z} = \frac{b}{a}$, and

$$a \sin. z - b \cos. z = 0 \qquad (2).$$

Eliminate a from the equations (1) and (2), and

$$b (\sin. z \cos. 45^0 + \cos. z . \sin. 45^0) = y \sin. 45^0 \sin. z,$$

$$\text{or } b . \sin. (45^0 + z) = y \sin. 45^0 \sin. z,$$

$$\text{and} \therefore y, \text{ or } a + b = \frac{b . \sin. (45^0 + z)}{\sin. 45^0 . \sin. z}.$$

Again, in order to compute the logarithm of $a + b + c$,

$$\text{since } a + b + c = y + c, \text{ make } \tan. z' = \frac{c}{y}$$

$$= \frac{c}{b} . \frac{\sin. 45^0 . \sin. z}{\sin. (45^0 + z)}, \text{ then, as before,}$$

$$a + b + c = c . \frac{\sin. (45^0 + z')}{\sin. 45^0 . \sin. z'},$$

and so on for $a + b + c + d$, &c.

2. It is required to compute the sine of an angle; for instance, the sine of 3^0.

$$\text{Sin. } 3^0 = \sin. (2^0 + 1^0) = \sin. 2^0 \cos. 1^0 + \cos. 2^0 \sin. 1^0,$$

$$= \sin. 2^0 \cos. 1^0 \left(1 + \frac{\sin. 1^0 \cos. 2^0}{\cos. 1^0 \sin. 2^0}\right)$$

$$= \sin. 2^0 \cos. 1^0 \left(1 + \frac{\tan. 1^0}{\tan. 2^0}\right).$$

Assume $\therefore \tan.^2 \theta = \frac{\tan. 1^0}{\tan. 2^0}$, and we shall have, in a logarithmic form,

$$\log.\sin.3^{\circ} = \log.\sin.2^{\circ} + \log.\cos.1^{\circ} + 2\log.\sec.\theta - 30.$$

3. Let it be required to compute a quantity P, such, that $P = (1+e')(1+e'')(1+e''') \times$ &c.; the law of the formation of e', e'', e''', &c. being

$$e' = \frac{1-\surd(1-e^2)}{1+\surd(1-e^2)},\ e'' = \frac{1-\surd(1-e'^2)}{1+\surd(1-e'^2)},\ e''' = \frac{1-\surd(1-e''^2)}{1+\surd(1-e''^2)}.$$

The computation is conveniently effected, thus, put $e = \sin.\theta$;

$$\therefore\ \surd(1-e^2) = \cos.\theta, \text{ and } e' = \frac{1-\cos.\theta}{1+\cos.\theta} = \text{[page 33] } \tan.^2\frac{\theta}{2}$$

$$\text{and, } 1+e' = 1+\tan.^2\frac{\theta}{2} = \sec.^2\frac{\theta}{2}.$$

Again, put $e' = \sin.\theta'$;

$$\therefore\ e'' = \frac{1-\cos.\theta'}{1+\cos.\theta'} = \tan.^2\frac{\theta'}{2} \text{ and } 1+e'' = \sec.^2\frac{\theta'}{2}.$$

Again, put $e'' = \sin.\theta''$;

$$\therefore\ e''' = \frac{1-\cos.\theta''}{1+\cos.\theta''} = \tan.^2\frac{\theta''}{2}, \text{ and } 1+e''' = \sec.^2\frac{\theta''}{2}, \text{ \&c.}$$

$$\text{Hence } P = \sec.^2\frac{\theta}{2}.\sec.^2\frac{\theta'}{2}.\sec.^2\frac{\theta''}{2}. \text{ \&c.}$$

and log. P = [supplying the tabular radius]

$$2\left(\log.\sec.\frac{\theta}{2} + \log.\sec.\frac{\theta'}{2} + \log.\sec.\frac{\theta''}{2} + \text{\&c.}\right)$$

$$-\ 2(10+10+10+\text{\&c.})$$

4. *Required the integrals of the differential expressions, $d\theta.\sin.\theta.\cos.\theta$; $d\theta.\cos.^2\theta$; $d\theta.\sin.^2\theta$; $d\theta\cos.^3\theta$; $d\theta.\cos.\theta\cos.n\theta$; $\dfrac{d\theta}{\cos.^4\theta}$.

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* This, in other words and symbols, is to require the fluents of the fluxionary expressions $\dot{\theta}.\sin.\theta\cos.\theta$; $\dot{\theta}\cos.^2\theta$, &c.

In order to integrate these expressions, it is necessary to premise (see Simpson's *Fluxions*, Vol. I, p. 165.) θ being any arc, that

$$d\theta = \frac{d(\sin.\theta)}{\cos.\theta}, \text{ or } = -\frac{d(\cos.\theta)}{\sin.\theta}, \text{ or } = \frac{d(\tan.\theta)}{\sec.^2\theta}.$$

1st. $\int d\theta . \cos.\theta . \sin.\theta.$

By the formula of p. 42,

$$\cos.\theta . \sin.\theta = \frac{1}{2} . \sin. 2\theta;$$

$$\therefore \int d\theta . \cos.\theta . \sin.\theta = \frac{1}{2}\int d\theta . \sin. 2\theta = \frac{1}{2}\int 2\, d\theta . \sin. 2\theta$$

$$= -\frac{1}{4}\cos. 2\theta + c \quad (c = \text{correction}).$$

2d. $\int d\theta . \cos.^2\theta.$

By the formula of p. 42,

$$\cos.^2\theta = \frac{1}{2}(1 + \cos. 2\theta;)$$

$$\therefore \int d\theta . \cos.^2\theta = \int \frac{d\theta}{2} + \frac{1}{2}\int d\theta . \cos. 2\theta$$

$$= \int \frac{d\theta}{2} + \frac{1}{4}\int 2\, d\theta . \cos. 2\theta$$

$$= \frac{\theta}{2} + \frac{1}{4}\sin. 2\theta + c.$$

3d. $\int d\theta . \sin.^2\theta.$

By the formula of p. 42,

$$\sin.^2\theta = \frac{1}{2}(1 - \cos. 2\theta);$$

$$\therefore \int d\theta . \sin.^2\theta = \int \frac{d\theta}{2} - \frac{1}{2}\int d\theta . \cos. 2\theta$$

$$=\int\frac{d\theta}{2}-\frac{1}{4}\int 2\,d\theta\,.\cos.\,2\,\theta$$

$$=\frac{\theta}{2}-\frac{1}{4}\sin.\,2\,\theta+c.$$

4th. $\int d\theta\,.\cos.^3\,\theta.$

By the formula of p. 60,

$$\cos.^3\,\theta=\frac{1}{4}\cos.\,3\,\theta+\frac{3}{4}\cos.\,\theta\,;$$

$$\therefore\int d\theta\cos.^3\,\theta=\frac{1}{4}\int d\theta\,.\cos.\,3\,\theta+\frac{3}{4}\int d\theta\,.\cos.\,\theta$$

$$=\frac{1}{3\,.\,4}\int 3\,d\theta\,.\cos.\,3\,\theta+\frac{3}{4}\sin.\,\theta$$

$$=\frac{1}{3\,.\,4}\,.\sin.\,3\,\theta+\frac{3}{4}\sin.\,\theta+c.$$

5th. $\int d\theta\,.\cos.\,\theta\,.\cos.\,n\,\theta.$

By the formula of p. 45, l. 16,

$$\cos.\,\theta.\cos.\,n\,\theta=\frac{1}{2}\left\{\cos.\,(n-1)\,\theta+\cos.\,(n+1)\,\theta\right\};$$

$$\therefore\int d\theta\,.\cos.\,\theta\cos.\,n\,\theta=\frac{1}{2}\int d\theta\,.\cos.\,(n-1)\,\theta+\frac{1}{2}\int d\theta\,.\cos.\,(n+1)\,\theta$$

$$=\frac{1}{2\,.\,(n-1)}\int(n-1)\,d\theta\,.\cos.\,(n-1)\,\theta$$

$$+\frac{1}{2\,.\,(n+1)}\int(n+1)\,d\theta\cos.\,(n+1)\,\theta=$$

$$\frac{1}{2\,.\,(n-1)}\sin.\,(n-1)\,\theta+\frac{1}{2\,(n+1)}\sin.\,(n+1)\,\theta+c.$$

6th. $\int\frac{d\theta}{\cos.^4\,\theta}.$

By p. 8, $\frac{1}{\cos.^4\,\theta}=\sec.^4\,\theta=(\sec.^2\,\theta)^2=(1+\tan.^2\,\theta)^2\,;$

$$\therefore \int \frac{d\theta}{\cos.^4 \theta} = \int d\theta\,(1 + \tan.^2 \theta)^2 = \int d\theta\,(1 + \tan.^2 \theta) \times (1 + \tan.^2 \theta)$$

$$= \int \{ d(\tan.\theta)(1 + \tan.^2 \theta) \} = \tan.\theta + \frac{1}{3}\tan.^3 \theta + c.$$

5. If $\frac{1}{y} = \frac{1 - \epsilon\cos.\varsigma}{1 - \epsilon^2}$, it is required to express $\frac{1}{y^2}$, $\frac{1}{y^3}$, &c. in terms involving the cosines of multiples of the arc ς; see Mayer's *Theory of the Moon*, p. 14.

$$\frac{1}{y} = \frac{1 - \epsilon.\cos.\varsigma}{1 - \epsilon^2} = 1 + \epsilon^2 - \epsilon.\cos.\varsigma,$$

rejecting* the terms involving ϵ^3, ϵ^4, &c.

$$\therefore \frac{1}{y^2} = (1+\epsilon^2)^2 - 2.(1+\epsilon^2)\epsilon.\cos.\varsigma + \epsilon^2.\cos.^2\varsigma,$$

$$= 1 + 2\epsilon^2 - 2\epsilon.\cos.\varsigma + \epsilon^2 \left\{\frac{1}{2} + \frac{\cos.2\varsigma}{2}\right\}$$

$$= 1 + \frac{5\epsilon^2}{2} - 2\epsilon.\cos.\varsigma + \frac{\epsilon^2}{2}\cos.2\varsigma.$$

Again,

$$\frac{1}{y^3} = (1 + \epsilon^2)^3 - 3\epsilon.(1 + \epsilon^2)^2\cos.\varsigma + 3\epsilon^2(1 + \epsilon^2)\cos.^2\varsigma$$

$$= 1 + 3\epsilon^2 - 3\epsilon.\cos.\varsigma + 3\epsilon^2\left\{\frac{1}{2} + \frac{\cos.2\varsigma}{2}\right\}.$$

$$= 1 + \frac{9\epsilon^2}{2} - 3\epsilon.\cos.\varsigma + \frac{3\epsilon}{2}\cos.2\varsigma,$$

and generally

$$\frac{1}{y^n} = (1+\epsilon^2)^n - n(1 + \epsilon^2)^{n-1}\epsilon.\cos.\varsigma + n.\frac{n-1}{2}(1 + \epsilon^2)^{n-2}\epsilon^2.\cos.^2\varsigma$$

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* The rejection of the terms involving ϵ^3, ϵ^4, &c. is not, as it is plain, essential to the illustration of the use of Trigonometrical formulæ, but, we have given the instance as Mayer has, and as all similar instances in Physical Astronomy will be circumstanced, in which ϵ denotes the eccentricity of an orbit.

$$= 1 + n\epsilon^2 - n\epsilon . \cos . \varsigma + n . \frac{n-1}{2}\epsilon^2 \left\{ \frac{1}{2} + \frac{\cos . 2\varsigma}{2} \right\}$$

$$= 1 + \frac{n . n+3}{4}\epsilon^2 - n\epsilon . \cos . \varsigma + \frac{n . n-1}{4}\epsilon^2 \cos . 2\varsigma .$$

In this instance, it is plain, if all terms involving higher powers of ϵ than that of the square be rejected, that no higher multiple of ς than 2ς can be introduced. The quantity ς is the mean anomaly, and y is the radius vector.

6. If $s = \sqrt{(f^2 - 2fr . \cos . t + r^2)}$, it is required to express $\frac{1}{s^3}$ by a series of terms involving the cosine of the multiples of the arc t. (See *Mem. Acad. des Sciences,* 1754, p. 538. *Clairaut sur l'orbite apparent du Soleil*).

Let $2fr \cos . t - r^2 = a$, then

$s = \sqrt{(f^2 - a)}$, and by the binomial theorem,

$$\frac{1}{s^3} = \frac{1}{f^3} + \frac{3a}{2f^5} + \frac{15a^2}{8f^7} + \frac{35a^3}{16f^9} + \frac{315}{128} . \frac{a^4}{f^{11}} + \&c.$$

$$\begin{aligned} \text{But, } a^2 &= r^4 - 4fr^3 . \cos . t + 4f^2r^2 . \cos .^2 t, \\ \text{(by Prob. 6,)} \quad &= r^4 - 4fr^3 . \cos . t + 2f^2r^2 (1 + \cos . 2t) \\ &= r^4 + 2f^2r^2 - 4fr^3 \cos . t + 2f^2r^2 . \cos . 2t . \end{aligned}$$

Again,

$$\begin{aligned} a^3 &= -r^6 + 6fr^5 \cos . t - 12f^2r^4 \cos .^2 t + 8f^3r^3 \cos .^3 t \\ &= -r^6 - 6f^2r^4 + (6fr^5 + 6f^3r^3) \cos . t - \\ &\quad 6f^2r^4 . \cos . 2t + 2f^3r^3 \cos . 3t . \quad \text{(Prob. 7, } [e''']) \end{aligned}$$

Substituting these values in the series for $\frac{1}{s^3}$, there results

$$\begin{aligned} \frac{1}{s^3} = &\frac{1}{f^3} + \frac{9r^2}{4f^5} + \frac{225r^4}{64f^7} \\ &+ \left(\frac{3r}{f^4} + \frac{45r^3}{8f^6} \right) \cos . t \\ &+ \left(\frac{13r^2}{4f^5} + \frac{105r^4}{16f^7} \right) \cos . 2t + \frac{35r^3}{8f^6} \cos . 3t . \end{aligned}$$

In this instance, r is the radius of the Earth's orbit, and f that of Jupiter's, and since $\frac{r}{f}$ is a small fraction ($=.19245$) the terms involving $\frac{r^5}{f^8}$, $\frac{r^6}{f^9}$, &c. are rejected, (see *Astron.* vol. II. p. 283.)

The quantity $\frac{1}{s^3}$ is a factor in a term dependent, in the theory of a planet disturbed in its orbit, on the disturbing force; and, the object of the above resolution is to resolve the term into a series such as

$$A^{(0)} + A^{(1)}.\cos. t + A^{(2)} \cos. 2t + \&c.$$

which is easily effected when $\frac{r}{f}$ is a small fraction, and may be, but not without artifice, when $\frac{r}{f}$ is nearly equal 1, that is, when the radii of the disturbing and disturbed body are nearly equal. (See *Phil. Trans.* 1804, pp. 265, &c. *Mem. Acad.* 1754, p. 545, and *Astron.* vol. II. Chap. XVIII.)

7. If the terms of the series

$$A^{(0)} + A^{(1)} \cos. t + A^{(2)} \cos. 2t + \&c.$$

be multiplied by $\cos. mt$, it is required so to transform the terms that the series shall preserve its original form.

By the formula [d] of p. 32,

$$\cos. nt . \cos. mt = \frac{1}{2} . \{ \cos. (m-n)t + \cos. (m+n)t \}$$

Hence,

$$\cos. t . \cos. mt = \frac{1}{2} \{ \cos. (m-1)t + \cos. (m+1)t \}$$

$$\cos. 2t . \cos. mt = \frac{1}{2} \{ \cos. (m-2)t + \cos. (m+2)t \}$$

&c.

which values being substituted in

$$A^{(0)} . \cos. mt + A^{(1)} \cos. t . \cos. mt + A^{(2)} \cos. 2t . \cos. mt + \&c.$$

and the terms properly arranged, what is required will be done. (See Laplace, *Mem. Acad.* 1785, pp. 54, &c. and *Mecanique Celeste*, p. 263.)

8. Required the sin. t, cos. t, sin. $2\,t$, &c.

$$\text{when } t = n\,v - \frac{2\,e}{m}(1 - n)\,\text{sin. } m\,v\,;$$

the coefficient $\frac{2\,e}{m}(1 - n)$ being a very small quantity.

(See *Acad. des Sciences*, 1754, p. 539: also for similar instances, *Acad. des Sciences*, 1745, p. 348, and Clairaut's *Theory of the Moon*, p. 20, and *Astron.* vol. II. pp. 140, &c.)

By the formula [1], p. 27,

$$\text{sin. } t = \text{sin. } n\,v\,.\,\text{cos. } \left\{\frac{2\,e}{m}(1-n)\,\text{sin. } m\,v\right\}$$

$$- \text{cos. } n\,v\,.\,\text{sin. } \left\{\frac{2\,e}{m}(1-n)\,\text{sin. } m\,v\right\}.$$

Now the quantity* $\frac{2\,e}{m}(1-n)\,\text{sin. } mv$, by the hypothesis, is very small; therefore its cosine is nearly $= 1$, and its sine is nearly the arc which it is supposed to represent: consequently,

$$\text{sin. } t = \text{sin. } n\,v - \frac{2\,e}{m}(1-n)\,\text{sin. } m\,v\,.\,\text{cos. } n\,v,$$

$$[\text{by } (b) \text{ p. } 31,] = \text{sin. } n\,v - \frac{e}{m}(1-n)\,\{\,\text{sin. } (n+m)\,v - \text{sin. } (n-m)\,v\,\}$$

Again,

$$\text{cos. } t = \text{cos. } n\,v + \frac{2\,e}{m}(1 - n)\,\text{sin. } m\,v\,.\,\text{sin. } n\,v$$

$$[\text{by } (e) \text{ p. } 31,] = \text{cos. } n\,v + \frac{e}{m}(1-n)\,\{\,\text{cos. } (n-m)\,v - \text{cos. } (n+m)\,v\,\}$$

---

* As in the former instance, (see Note, p. 108,) so in this, the smallness of the quantity $\frac{2\,e}{m}(1-n)$ is, in no wise, essential to the illustration of the use of the Trigonometrical formulæ.

and, by a like process, may sin. $2t$, cos. $2t$, &c. be deduced.

9. Required the value of

* $\sin. v\int Q \cos. v \,.\, dv - \cos. v \int Q \sin. v \,.\, dv,$

when Q is represented by a series of terms

$$a \,.\, \cos. nv + b \,.\, \cos. mv + \&c.$$

(See Clairaut, *Acad. des Sciences*, 1745, p. 341, also his *Theorie de la Lune*, edit. 2. p. 9. Lalande, *Acad. des Sciences*, 1760, p. 313. Laplace, *Mec. Celeste*, p. 241. Thomas Simpson's *Tracts*, pp. 92, &c. Cousin's, *Physical Astronomy*, pp. 23, &c. Vince's *Astronomy*, Vol. II. pp. 168, &c. and Woodhouse's *Astron.* vol. II. pp. 99, &c.

If we substitute the first term $a \,.\, \cos. nv$ of the series in the expression, then

$$\sin. v \int Q \cos. v \,.\, dv = a \,.\, \sin. v \int \cos. v \,.\, \cos. nv \,.\, dv$$

$$[\text{form } (d) \text{ p. } 31,] = a \sin. v \frac{1}{2} \int \{ \cos. (n-1)v + \cos. (n+1)v \} dv$$

$$(\text{p. } 106,) = \frac{a}{2} \sin. v \left\{ \frac{\sin. (n-1)v}{n-1} + \frac{\sin. (n+1)v}{n+1} + C \right\}$$

where C, the correction, will $= 0$, if the integral $= 0$ when $v = 0$.

Again,

$$\cos. v \int Q \sin. v \,.\, dv = a \cos. v \int \sin. v \,.\, \cos. nv \,.\, dv,$$

$$(\text{form } [b] \text{ p. } 31,) = a \cos. v \tfrac{1}{2} \int \{ \sin. (n+1)v - \sin. (n-1)v \} dv$$

$$(\text{p. } 116,) = a \cos. v \left\{ \frac{\cos. (n-1)v}{2(n-1)} - \frac{\cos. (n+1)v}{2(n+1)} + C' \right\}$$

in which, according to the preceding hypothesis of the correction,

$$0 = \frac{1}{2 \,.\, (n-1)} - \frac{1}{2 \,.\, (n+1)} + C',$$

$$\text{and } C' = -\frac{n^2 - 1}{1} \,.$$

---

* dv is the differential of v answering to $\dot{v}$ the fluxion of v.

Hence, combining the two parts of the expression,

$$\sin.\, v\!\int\! Q.\cos.\, n\,v\,.\,d\,v - \cos.\, v\!\int\! Q \sin.\, n\,v\,.\,d\,v =$$

$$\frac{a}{2\,(n-1)}\left\{\sin.\,(n-1)\,v\,.\,\sin.\, v - \cos.\,(n-1)\,v\,.\cos.\, v\right\}$$

$$+\frac{a}{2.(n+1)}\left\{\cos.\,(n+1)\,v\,.\,\cos.\, v + \sin.\,(n+1)\,v\,\sin.\, v\right\}$$

$$+\frac{a\,.\cos.\, v}{n^2-1} \quad = \text{(by the forms [2] and [4] pp. 30, 31.)}$$

$$-\frac{a\,.\cos.\, n\,v}{2\,(n-1)}+\frac{a\,.\cos.\, n v}{2.(n+1)}+\frac{a\,.\cos.\, v}{n^2-1}=$$

$$\frac{a\,.\cos.\, v}{n^2-1}-\frac{a\,.\cos.\, n\,v}{n^2-1}\,.$$

If, instead of $a\,.\cos.\, n\,v$ for Q, we had substituted $b\,.\cos.\, mv$, the resulting value of the integral would have been

$$\frac{b\,.\cos.\, v}{m^2-1}-\frac{b\,.\cos.\, m\,v}{m^2-1}\,.$$

Hence, the whole integral, when Q is represented by the series $a\,.\cos.\, n\,v + b\,.\cos.\, m\,v + \&c.$ is equal

$$\left\{\frac{a}{n^2-1}+\frac{b}{m^2-1}+\&c.\right\}\cos.\, v$$

$$-\left(\frac{a\,.\cos.\, n\,v}{n^2-1}+\frac{b\,.\cos.\, m\,v}{m^2-1}+\&c.\right).$$

10. Expand (see Ex. 5.) $\frac{u'^3}{u^4}\,s$ into a series of cosines of arcs, when

$$u' = \frac{1}{a'}\,(1 + e' \cos.\, c' m\,v)$$

$$u = \frac{1}{a}\,(1 + e \cos.\, c\,v)$$

$$s = \gamma\,.\sin.\,(g\,v - \theta)$$

e' e, γ being very small quantities, (*see Astron.* vol. II. p. 247.)

1st, nearly, $u'^3 = \frac{1}{a'^3}(1 + 3e' \cos. c' m v)$

$$\frac{1}{u^4} = a^4 (1 - 4e \cos. c v);$$

$$\therefore \frac{u'^3}{u^4} = \frac{a^4}{a'^3}(1 - 4e . \cos. c v + 3e' \cos. c' m v)$$

but $\quad s = \gamma \sin. (g v - \theta);$

$$\therefore \frac{u'^3}{u^4} s = \frac{a^4}{a'^3} \gamma \left\{ \begin{array}{l} \sin. (g v - \theta) \\ - 2e . \sin. (g v + c v - \theta) \\ - 2e . \sin. (g v - c v - \theta) \\ + \frac{3e'}{2} . \sin. (g v + c' m v - \theta) \\ + \frac{3e'}{2} . \sin. (g v - c' m v - \theta) \end{array} \right\}$$

11. Required the value (I) of $\int dz . \cos. m z \times \cos. n z \times \cos. p z$. (See Simpson's *Tracts*, p. 89.)

$$\cos. m z \times \cos. n z \times \cos. p z = \frac{1}{2} \{ \cos. (m-n) z + \cos. (m+n) z \} \cos. p z$$

$$= \frac{1}{4} \{ \cos. (m-n-p) z + \cos. (m-n+p) z \}$$

$$+ \frac{1}{4} \{ \cos. (m+n-p) z + \cos. (m+n+p) z \}$$

consequently the integral (I) equals

$$\frac{\sin. (m-n-p) z}{4 . (m-n-p)} + \frac{\sin. (m-n+p)}{4 . (m-n+p)}$$

$$+ \frac{\sin. (m+n-p) z}{4 (m+n-p)} + \frac{\sin. (m+n+p) z}{4 (m+n+p)} .$$

12. Let it be required to compound (F),

$$\sin. (2 v - 2 \theta) + \sin. (2 v' - 2 \theta) - \sin. (2 v - 2 v')$$

into one term, the product of three sines or cosines (see *Astron.* vol. II. pp. 441, 442.)

By forms (5), (6), *Trig.* ed. 3. p. 32.

$$F = 2\sin.\ \{(v-\theta)+(v'-\theta)\}\ \{\cos.(v-v')\} - \sin.(2v-2v')$$

$$= 2\sin.\ \Big\{\{(v-\theta)+(v'-\theta)\} - \sin.(v-v')\Big\} \times \cos.(v-v')$$

$$= 4\cos.(v-\theta)\sin.(v'-\theta)\cos.(v-v')$$

This process is, in its nature, the reverse of that in the preceding example.

13. It is required to resolve $(r^2 - 2rr'\cos.\omega + r'^2)^{-m}$* (F) into a series such as

$$A + B.\cos.\omega + C.\cos.2\omega + D.\cos.3\omega + \&c.$$

(see *Astron.* vol. II. p. 260: also *Lacroix*, tom. II. p. 132. *Laplace*, tom. I. pp. 271, &c.)

Make $2\cos.\omega = x + \frac{1}{x}$, then

$$r^2 - 2rr'\cos.\omega + r'^2 = r^2 - r'r\left(x+\frac{1}{x}\right) + r'^2$$

$$= (r'-rx)\left(r'-\frac{r}{x}\right)$$

$$= r'^2.\left(1-\frac{r}{r'}x\right)\left(1-\frac{r}{r'}x\right);$$

$$\therefore F = r'^{-2m}.\left(1-\frac{r}{r'}x\right)^{-m}.\left(1-\frac{r}{r'}.\frac{1}{x}\right)^{-m}.$$

Now,

$$\left(1-\frac{r}{r'}x\right)^{-m} = 1 + \frac{mr}{r'}x + \frac{m.(m+1)}{1.2}\left(\frac{r}{r'}\right)^2 x^2$$

$$+ \frac{m(m+1)(m+2)}{1.2.3}\left(\frac{r}{r'}\right)^3 x^3 + \&c.$$

$$\left(1-\frac{r}{r'}\frac{1}{x}\right) = 1 + \frac{mr}{r'}.\frac{1}{x} + \frac{m.(m+1)}{1.2}\left(\frac{r}{r'}\right)^2.\frac{1}{x^2}$$

$$+ \frac{m(m+1)(m+2)}{1.2.3}\left(\frac{r}{r'}\right)^3\frac{1}{x^3} + \&c.$$

---

* See Examples 6 and 7.

$$\therefore A = \frac{1}{r'^{2m}} \left\{ 1 + \left(\frac{m\,r}{r'}\right)^2 + \left(\frac{m\,.\,m+1}{1\,.\,2}\right)^2 . \left(\frac{r}{r'}\right)^4 \right.$$

$$\left. + \left(\frac{m.(m+1)(m+2)}{1\,.\,2\,.\,3}\right)^2 \left(\frac{r}{r'}\right)^6 + \&c. \right.$$

$$B = \frac{2}{r'^{2m}} \left\{ \frac{m\,r}{r'} + \frac{m^2}{1^2} . \frac{m+1}{2} \left(\frac{r}{r'}\right)^3 \right.$$

$$\left. + \left(\frac{m\,.\,m+1}{1\,.\,2}\right)^2 . \frac{m+2}{3} \left(\frac{r}{r'}\right)^5 + \&c. \right\}$$

$$C = \frac{2}{r'^{2m}} \left(\frac{m\,.\,m+1}{1.2} . \frac{r^2}{r'^2} + \frac{m^2}{1^2} . \frac{m+1}{2} . \frac{m+2}{2} . \frac{r^4}{r'^4} + \&c. \right)$$

$$D = \&c.$$

14. In the Lunar Theory, one of the *equations* (that of the *evection*) is represented by $-\,2\,r\,.\,\sin.\,(2\,X - Z,)$ r being supposed to be of a mean constant value: if r should vary and proportionally to $1 - 3\,c\,.\,\cos.\,Y,$ what is the new *equation* that would thence arise? (See Vince's *Astronomy*, vol. II. p. 53.)

Let m be the mean value of r, then

$$\text{the } \textit{evection} = ,-\,2\,m\,(1 - 3\,c\,.\cos.\,Y)\sin.\,(2\,X - Z)$$

$$= -\,2\,m\,\sin.\,(2X - Z) + 6\,m\,c\,\cos.\,Y.\sin.\,(2X - Z.)$$

Therefore, since the first term denotes the mean value, the second, that is, $6\,m\,c\,\cos.\,Y\,\sin.\,(2\,X - Z)$, denotes the variation from that mean value, or the *new equation*.

Now, $6\,m\,c\,.\cos.\,Y.\sin.\,(2\,X - Z) =$ (by form [*a*] p. 26.)

$$3\,m\,c\,\{\sin.\,(2\,X - Z + Y) + \sin.\,(2\,X - Z - Y)\}$$

which is the form of the new *equation*, in which, the angles $2\,X - Z + Y$, $2\,X - Z - Y$, are technically denominated the *arguments*, and $3\,m\,c$ ($= 2'\ 1''$) is the *coefficient*.

The Student, perhaps, may now be inclined to believe that the formulæ demonstrated in the preceding pages, are not entirely without their use, nor invented and shewn as mere specimens of analytical dexterity. The instances, indeed, have been, almost all, taken from Tracts on Physical Astronomy; and it is, therefore, on the assumption of the utility of that science, that the Trigonometrical analysis has been affirmed to be useful.

We will proceed to apply the formulæ of Trigonometry to the resolution of certain numerical equations: an application less extensive than the preceding, and of more doubtful utility.

CHAP. VII.

On the Solution of certain Numerical Equations by means of Trigonometrical Formulæ and Tables.

In this application of the Trigonometrical Analysis, the utility, whatever it be, relates to the expedition and convenience of the resolution of the equations, and not to any thing novel or curious in its principle and method; moreover, the expedition and conciseness of the resolution depends not on any essential and real abridgment, but on that sort and kind of abridgment which registered computations or tables afford: for instance, we shewed (page 68.) that a cubic equation, such as

$$\text{sin. } 3A = 3 \text{ sin. } A - 4 \text{ sin.}^3 A$$

when solved by approximation, might be used for the computation of sines, or the construction of Tables. Reversely, the Tables constructed may be used for the resolution of similar cubic equations; again, we shewed (page 68.) that an equation of 5 dimensions, such as

$$2 \text{ sin. } 5A, \text{ or } s = 5x - 5x^3 + x^5$$

solved by approximation, might be made subservient to the construction of Tables. Reversely, Tables constructed either by the approximate solution of equations, or by other methods of approximation, may be employed in the numerical resolution of similar equations. De Moivre solved equations of the third and fifth degree by the cosines of the third part and fifth of an arc; and Vieta divided an arc into three and five equal parts, by equations of the third and fifth degree. This is sufficient, perhaps, to explain the real principle of the solution of equations by Trigonometrical Tables. The convenience or expedition of the method, as we have already said, is of the same nature as the expedition of computation by logarithms. If we do not avail ourselves of computed Tables, the whole process of the solution of a cubic equation, for instance, will be tedious; we must employ some method of approximation ; now, if Trigonometrical Tables are employed, the process is short and easy, but only so, because the most laborious part of it is already done for us. In these cases, we avail ourselves of the registered computations of preceding mathematicians.

Solution of a Cubic Equation.

It may be proper, however, to shew more in detail the method of solving equations by the aid of Trigonometrical Tables.

If we take the equations [c'''] and [s'''] pp. 45, 47, supply a radius r, and put c and s for cos. 3 A, and sin. 3 A, respectively, and x for cos. A and sin. A, then

$$c\,r^2 = 4\,x^3 - 3\,r^2\,x, \text{ or } x^3 - \frac{3\,r^2}{4}\,x - \frac{c\,r^2}{4} = 0 \ldots\ldots\ldots [1]$$

$$\text{and } s\,r^2 = 3r^2x - 4\,x^3, \text{ or, } x^3 - \frac{3\,r^2}{4}\,x + \frac{s\,r^2}{4} = 0 \ldots\ldots\ldots [2]$$

Hence, c being given, find in the Tables the arc 3 A corresponding to it, and, from the same Tables, take out the cosine of A ; this latter value is the root of the equation [1]. If s be given, find 3 A corresponding to the sine s, and then take from the Tables the sine of A, which is the root of the equation [2].

But, cubic equations have 3 roots; now, by the Table of p. 16, or by the Cor. 6. of page 28, if c = cos. 3 A, then also c = cos. $(2\pi - 3A)$ = cos. $(2\pi + 3A)$ = cos. $(4\pi - 3A)$, &c. hence, substituting instead of the arc A, the arcs $\frac{2\pi}{3} - A$, $\frac{2\pi}{3} + A$, $\frac{4\pi}{3} - A$, &c. any and all, of the following equations, are true.

$$c\,r^2 = 4.\left(\cos.\ \frac{2\pi}{3} - A\right)^3 - 3\,.\,r^2\,.\cos.\left(\frac{2\pi}{3} - A\right) \ldots\ldots [a]$$

$$c\,r^2 = 4.\left(\cos.\ \frac{2\pi}{3} + A\right)^3 - 3\,.\,r^2\,.\cos.\left(\frac{2\pi}{3} + A\right) \ldots\ldots [b]$$

$$c\,r^2 = 4.\left(\cos.\ \frac{4\pi}{3} - A\right)^3 - 3\,.\,r^2\,.\cos.\left(\frac{4\pi}{3} - A\right) \ldots\ldots [d]$$

&c.

But, $\cos.\left(\frac{4\pi}{3} - A\right) = \cos.\left\{2\pi - \left(\frac{2\pi}{3} + A\right)\right\} =$ $\cos. 2\pi . \cos.\left(\frac{2\pi}{3} + A\right) = \cos.\left(\frac{2\pi}{3} + A\right)$ [since $\sin. 2\pi = 0$, and $\cos. 2\pi = 1$]. Hence, the equation [*d*] is precisely the same as the equation [*b*].

Again, if we take the equation that would follow the equation [*d*], the cosine in it would be $\cos.\left(\frac{4\pi}{3} + A\right)$; now, $\cos.\left(\frac{4\pi}{3} + A\right)$ $= \cos.\left\{2\pi - \left(\frac{2\pi}{3} - A\right)\right\} = \cos. 2\pi . \cos.\left(\frac{2\pi}{3} - A\right) =$ $\cos.\left(\frac{2\pi}{3} - A\right)$: hence, this equation is precisely the same as the equation [*a*]; and, in like manner, succeeding equations would recur, so that, essentially, there are only 3 different equations [1], [*a*], {*b*], and hence, in the equation

$$c r^2 = 4x^3 - 3r^2 x, \text{ or, } x^3 - \frac{3r^2}{4}x - \frac{cr^3}{4} = 0,$$

x may, either, $= \cos. A$, or, $= \cos.\left(\frac{2\pi}{3} - A\right)$, or, $=$ $\cos.\left(\frac{2\pi}{3} + A\right)$, c being the cosine of $3A$.

The same reasoning applies to the equation [2], and similar conclusions will follow.

Example.

Let $x^3 - 147x - 285.5 = 0$; compare this with

$$x^3 - \frac{3r^2}{4}x - \frac{cr^2}{4} = 0;$$

$\therefore \frac{3r^2}{4} = 147$, and $r = 14$: and $c = \frac{285.5}{49}$ and the tabular cosine $=$

$$10000\,\frac{285.5}{49 \times 14}$$

$= 416.808$, &c. Now the arc corresponding to this cosine, or $3A$, is $65° 24'$;

$\therefore A = 21°\,48'$ nat. cos. $= .9284858$ $\therefore$ to rad. $14 = 12.9988012$, 1st root.

Again,

$$120° - 21°.48' = 98°.12', \text{ nat. cos. } = -.1426289;$$
$$\therefore \text{ to rad. } 14 = -1.9968046, \text{ 2d root.}$$

Again,

$$120° + 21°\,48' = 141°.48', \text{ nat. cos. } = -.7858569;$$
$$\therefore \text{ to rad. } 14 = -11.0019966, \text{ 3d root,}$$

and the sum of these two last roots, that is, of the 2d and 3d roots, exactly equals the first root, which ought to be the case, since the coefficient of the 2d term in the proposed equation is $= 0$.

But, cubic equations of any form may be solved by substituting Trigonometrical lines, sines, or tangents, for the quantities under the radical sign in Cardan's form: as Dr. Maskelyne has done in page 57 of his Introduction to Taylor's *Logarithms*.

Not only cubic, but quadratic equations may be solved by the aid of Trigonometrical Tables; and conveniently, when the coefficients are expressed by many figures.

Solution of a Quadratic Equation.

Let $x^2 + px - q = 0$,

then $x = -\frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} + q\right)} = -\frac{p}{2}\left\{1 \mp \sqrt{\left(1 + \frac{4q}{p^2}\right)}\right\}$.

Assume $\frac{4q}{p^2} = \tan.^2 \theta$, then, $x = -\frac{p}{2}(1 \mp \sec. \theta)$.

Now by Prob. 3. Cor. 5, pp. 33, 34.

$$\cos. \theta = \frac{1 - \tan.^2 \frac{\theta}{2}}{1 + \tan.^2 \frac{\theta}{2}}$$

$$\therefore \sec. \theta = \frac{1 + \tan.^2 \frac{\theta}{2}}{1 - \tan.^2 \frac{\theta}{2}}$$

$$1 - \sec.\theta = -\frac{2\tan.^2\frac{\theta}{2}}{1-\tan.^2\frac{\theta}{2}},\quad 1+\sec.\theta = \frac{2}{1-\tan.^2\frac{\theta}{2}}$$

But by Prob. 8, p. 62,

$$\tan.\theta = \frac{2\tan.\frac{\theta}{2}}{1-\tan.^2\frac{\theta}{2}};$$

$$\therefore\ 1-\sec.\theta = -\tan.\theta\,.\tan.\frac{\theta}{2},$$

$$\text{and } 1+\sec.\theta = \frac{\tan\theta}{\tan.\frac{\theta}{2}} = \tan.\theta\,.\cot.\frac{\theta}{2}.$$

$$\text{Hence, one root } (\alpha) = \frac{p}{2}\tan.\theta\,.\tan.\frac{\theta}{2},$$

$$\text{the other } (\beta) = -\frac{p}{2}\tan.\theta\,.\cot.\frac{\theta}{2}.$$

And the whole formula of solution, in logarithms, is

$$\log.\tan.\theta = \tfrac{1}{2}(20 + \log.4 + \log.q - 2\log.p),$$

$$\text{and } \log.\alpha = \log.\frac{p}{2} + \log.\tan.\theta + \log.\tan.\frac{\theta}{2} - 20,$$

$$\text{and for 2d root, } \log.\beta = \log.\frac{p}{2} + \log.\tan.\theta + \log.\cot.\frac{\theta}{2} - 20.$$

The solution may be exhibited under a different form, thus:

$$x = \frac{p}{2}\left\{1 \pm \sqrt{\left(1+\frac{4q}{p^2}\right)}\right\} = -\frac{p}{2}\left(1 \mp \frac{1}{\cos.\theta}\right)$$

$$= -\sqrt{q}\left(\frac{\cos.\theta \mp 1}{\tan.\theta\,.\cos.\theta}\right) = -\sqrt{q}\left(\frac{\cos.\theta \mp 1}{\sin.\theta}\right)$$

$$= \text{(with the lower sign)} - \sqrt{q} \times \cot.\frac{\theta}{2},\ \text{and}$$

$$= \text{(with the upper sign)}\quad \sqrt{q}\,.\tan.\frac{\theta}{2}.$$

Example. Let $x^2 + 13.56x - 72.31 = 0$.

Computation of θ.

10.. $= 10$
log. 2.. $= .3010300$
$\frac{1}{2}$ log. 72.31.. $= .9295992$
11.2306292
log. 13.56.. $= 1.1322597$
log. tan. θ.. $= 10.0983695$

$$\therefore\ \theta = 51^\circ . 26' . 2'',$$

$$\text{and } \frac{\theta}{2} = 25 . 43 . 1$$

Computation of positive root.	Computation of negative root.
log. 6.78 = .8312297	log. 6.78 = .8312297
log. tan. 51° 26′ 2″ = 10.0983695	log. tan. 51°.26′.2″ = 10.0983695
log. tan. 25 43 1 = 9.6827151	log. cot. 25.43.1 = 10.3172849
20.6123143	21,2468841
$\therefore$ log. x = .6123143	$\therefore$ log. $(-x)$ = 1.2468841
and x, or α = 4.09557	and $-x$, or $-\beta$ = 17.6556

Hence the two roots are 4.09557 and − 17.6556, and the sum of these two roots is − 13.56, the coefficient of the second term of the equation, as it ought to be.

The equation that has been solved is $x^2 + px - q = 0$; if it had been $x^2 - px - q = 0$, the 2 roots would have been

$$\frac{p}{2}\left\{1 + \sqrt{\left(1 + \frac{4q}{p^2}\right)}\right\}, \frac{p}{2}\left\{1 - \sqrt{\left(1 + \frac{4q}{p^2}\right)}\right\}$$

that is, the 2 roots of the former equation taken negatively.

If the equation to be solved be $x^2 - px + q = 0$, then

$$x = \frac{p}{2}\left\{1 \pm \sqrt{\left(1 - \frac{4q}{p^2}\right)}\right\}; \text{ assume } \frac{4q}{p^2} = \sin.^2 \theta$$

and $x = \frac{p}{2}(1 \pm \cos.\theta) = p . \cos.^2 \frac{\theta}{2}$, or $= p \sin.^2 \frac{\theta}{2}$.

Hence the rule of computation, logarithmically expressed, is $\log. \sin. \theta = \frac{1}{2}(20 + 2 \log. 2 + \log. q - 2 \log. p)$ and $\log. x = \log. p + 2 \log. \cos. \frac{\theta}{2} - 20$, or $= \log. p + 2 \log. \sin. \frac{\theta}{2} - 20$.

If the equation be $x^2 + p x + q = 0$, its roots are the negatives of the roots of the preceding equation.

The preceding solutions are, in fact, the same as those given by Dr. Maskelyne, page 56 in the Introduction to Taylor's *Logarithms*.

If, in mathematical researches, equations, like those that have been given of the second and third degree, presented themselves to be solved, their solution would be conveniently effected by the preceding methods, and by the aid of the Trigonometrical Tables; but, the truth is, in the application of Mathematics to Physics, the solution of equations is an operation that very rarely is requisite, and consequently the preceding application of Trigonometrical formulæ is to be considered as a matter rather of curiosity than of utility.

SPHERICAL TRIGONOMETRY.

CHAP. VIII.

Definitions.

1. A SPHERE is a solid terminated by a curve surface, of which all the points are equally distant from an interior point, called the centre of the sphere.

The surface of a sphere may be conceived to be generated by the revolution of a semicircle round its diameter.

2. Every section of a sphere, made by a plane, (so it will be demonstrated) is a circle. A *great* circle is that, the plane of which passes through the centre; a *small* circle, that, the plane of which does not pass through the centre.

3. The pole of a circle of a sphere, is a point in the surface, equally distant from every point of the circumference of the circle.

4. A plane is said to be a tangent of a sphere, when it has one point only common with the sphere.

5. A spherical triangle is a portion of the surface of a sphere included within three arcs* of three great circles, which arcs are called sides of the triangle.

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* Each arc is supposed to be less than a semicircle, for the properties of a triangle that has its sides a, b, and $c = \pi + x$ are always known from those of a triangle that has its sides a, b, and a third side $= \pi - x$.

6. The angles of a spherical triangle, are the angles contained between the planes in which the arcs or sides lie. (See *Euclid*, Book XI. Def. 6.)

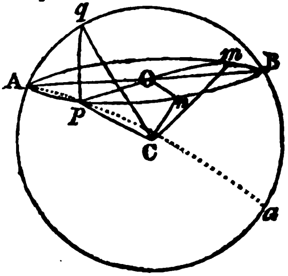
A spherical triangle is called rectangular, isosceles, equilateral, in the same cases that a plane triangle is.

◆

Proposition I.

Every section of a sphere made by a plane is a circle.

7.



Let $AnBm$ be the plane, draw CO* perpendicular to it, which consequently, (by *Euclid*, Book XI. Def. 3.) is perpendicular to every straight line meeting it in that plane; hence, since

$$Cn = Cm, \text{ and, } \angle Con = \angle Com = 90^{0}.$$

$$On^2 = Cn^2 - CO^2,\ Om^2 = Cm^2 - CO^2;\ \therefore On = Om,$$

and similarly, $On = Op$; $\therefore$ Om, On, Op are equal; $\therefore$ $ApBm$ is a circle, and O is its centre.

Cor. 1. When $CO = 0$, or when the plane passes through the centre C of the sphere, $On = Om = Cm = CB$, the radius of the sphere.

---

* CO is not drawn in the diagram.

Cor. 2. Hence two great circles always bisect each other; for, their common intersection passing through the centre is a diameter.

Cor. 3. Through two points on the surface of a sphere, such as A, p, a great circle, (part of which is represented in the Figure by the dotted curve from A to p) may be made to pass; for, the two points A, p, with C the centre as a third point, determine the position of a plane, the intersection of which with the surface of a sphere is a great circle passing through the points A, p. In like manner a great circle may pass through the points p, q.

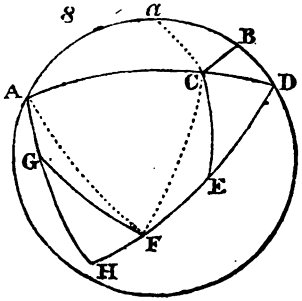
If the two points, instead of being A, p, be A, a, the two ends of a diameter, then the three points A, C, a lying in the same line, do not determine the position of a plane; but, through these three points, innumerable planes may pass.

Prop. II. In every spherical triangle as Apq, (Fig. p. 126.) any one side is less than the sum of the two others.

For the arcs Ap, Aq, pq, measure the angles ACp, ACq, pCq; but by *Euclid*, Book XI. Prop. 20, any angle, as ACp *, is less than the two others ACq, pCq; $\therefore$ Ap is less than $Aq + pq$.

Prop. III. The sum of the three sides of a spherical triangle is less than the circumference of a great circle (2π).

Let ACB be the spherical triangle; then, $CB < CD + BD$



by the former Proposition, and $AC + AB = AC + AB$; $\therefore$

---

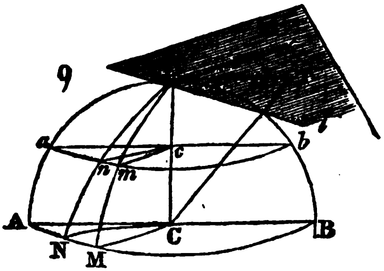
* Conceive, in Fig. p. 126, a straight line to be drawn from A to the centre C.

$AC + CB + AB < (AC + CD) + (BD + AB)$; $\therefore$ $< ACD + ABD$; $\therefore$ $< (\pi + \pi)$; $\therefore$ $< 2\pi$.

Cor. Hence, the sum of the sides of a polygon $ACEFGA$ (the sides being arcs of great circles) is less than 2π.

For, by the Proposition, $AD + DH + AH$ is $< 2\pi$;
or, $AC + (CD + DE) + EF + (FH + HG) + AG < 2\pi$,
but $CD + DE > CE$, and, $FH + HG > FG$;
$\therefore$ *a fortiori*, $AC + CE + EF + FG + AG < 2\pi$.

Prop. IV. If CD be drawn from the centre C, perpendicularly to the plane of the circle $ANMB$, then, D is the pole of the circle $ANMB$ and of all small circles, such as $anmb$, that are parallel to it.



For, since DC is perpendicular to the plane of $ANMB$, it is perpendicular to all lines in it, as CN, CM, &c. (See *Euclid*, Book XI. Def. 3). Hence, DCA, DCN, DCM, each $= 90^0$; consequently, since DC is common, and CA, CN, CM, are equal, the hypothenuses, which are the chords of the arcs DA, DN, DM, are equal; $\because$ the arcs DA, DN, DM are equal to one another and to 90^0, and $\therefore$ D, (by Def. 3.) is the pole of $ANMB$.

Again, in the small circle $anmb$, ca, cn, cm, are equal, as are the angles Dca, Dcn, Dcm; $\therefore$ as before the chords of Da, Dn, Dm, are equal, and the arcs Da, Dn, Dm; $\therefore$ D is equally distant from every point in the circumference of $anmb$, and therefore is its pole.

Cor. 1. By Definition 6, the spherical angle *AMD* is equal to the inclination of the planes *AMBC*, *DCM*, and therefore is a right angle.

Cor. 2. Hence, to find the pole of a great circle, draw, from the point *M*, a great circle perpendicular to *AM*, and take $MD = 90^\circ$; then *D* is the pole: and, reversely, if *D* be the pole, *DMN*, *DNM* are right angles, and *DN*, *DM* are quadrants.

Cor. 3. Hence, to describe a great circle, of which *D* is the pole, take *DN*, *DM*, each $= 90^\circ$; then let a plane pass through *N*, *M*, and *C*, and its intersection with the surface of the sphere is the circle required.

Cor. 4. Since *NDM* may be of any magnitude from 0 to 180°, and since the angles *DNM*, *DMN*, are, each $= 90^\circ$, the sum of the three angles of a triangle, as *DNM*, may be any angle between 180° and 360°, which are the two limits.

Cor. 5. The radius of the small circle *anmb* is *Cn*, which is the sin *Dn*, or, cos. *Nn*; and, if the great circle *ANMB* be divided into any number of equal parts, each equal to *NM*, the small circle *anmb* will be divided into the same number of equal parts, each part being equal to *nm*; but, the magnitude of *nm* will be to the magnitude of *NM*, as the circumference of *anmb* to that of *ANMB*, consequently, as the radius *cn* to the radius *CN*; or, as sin. *Dn* to sin. *DN*; as sin. *Dn* to radius; or, as cos. *Nn* to radius.

Prop. V. If a plane *TDl* is perpendicular to *CD*, it is a tangent to the sphere.

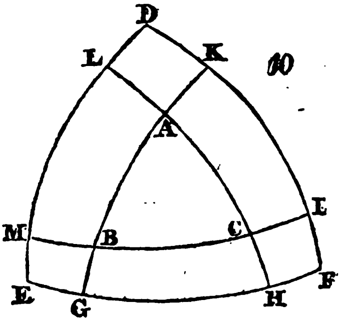
For take any point *T*, join *DT*, then *CDT* is a right angle; ∴ *CT* is greater than *CD*; ∴ *T* is without the surface, and since this is true of every point in the plane, except the point *D*, the plane *TDl* (by Def. 4.) is a tangent to the sphere.

Prop. VI. If *DT*, *Dt* be drawn, in the planes *DCA*, *DCN* respectively, tangents to the arcs *Da*, *Dn*, at the point *D*, the angle *TDt* is equal to the angle *ADN*, both of which angles are measured by the arc *AN*.

For, DT, Dt are perpendicular to DC, which is the intersection of the planes DCA, DCN; $\therefore$ by *Euclid*, Book XI. Def. 6. TDt measures the inclination of the planes, and therefore is equal to the angle ADN; but TDt, the inclination of the planes, $= ACN$, of which AN is the measure; $\therefore$ AN is the measure of TDt and of ADN.

Cor. If two arcs of great circles intersect each other, their vertical angles are equal.

Prop. VII. If from the points A, B, C, of the triangle ABC, as poles, be described, the arcs EF, FD, DE, forming a triangle DEF; then, reciprocally, the points D, E, F are poles of the arcs BC, AC, AB.



Since A is the pole of EF, the arc of a great circle drawn from A, to any point in EF, and therefore to a point as E, is a quadrant; similarly, since C is the pole of DE, the arc of a great circle from C, to any point in DE, and therefore to E, is a quadrant. Hence, E is distant from two points A, C, in an arc AC, by the quantity of a quadrant; $\therefore$ by Cor. 2, Prop. 4, E is the pole of AC; and similarly, F is the pole of AB, and D of BC.

Prop. VIII. The former construction remaining, the measures of the angles at A, B, C, are the supplements of the sides opposite, that is, of EF, DF, DE; and, reciprocally, the measures of the angles at D, E, F, are the supplements of the sides BC, AC, AB.

For, the measure of the angle at A (see Prop. 6.) is GH. Now, $GH = EH - EG = EH - (FE - FG) = EH + FG - FE = 90^\circ + 90^\circ - FE = 180^\circ - FE =$ (p. 12.) the supplement of FE.

Again,

the measure of the angle at B, or $KI = FK - FI = FK - (DF - DI)$ $= FK + DI - DF = 90^\circ + 90^\circ - DF = 180^\circ - DF = \text{supp}^t.\ DF$; and similarly, LM, the measure of the angle at C, is the supplement of DE.

Secondly, the measure of the angle at D, or $MI = MC + CI = MC + IB - BC = 90^\circ + 90^\circ - CB = 180^\circ - CB = \text{supp}^t.\ CB$; and similarly, LH, KG, the measures of the angles at E and F, are the supplements of AC and AB.

From its properties, the triangle DEF has been called, by English Geometers, the *Supplemental Triangle*; and from the mode of its description, by the French, the *Polar Triangle*.

Cor. 1. If any angle as B, $= 90^\circ$, then $DF = 180^\circ - 90^\circ = 90^\circ$, or is a quadrant: and if B and C each $= 90^\circ$, DF, DE, each is equal to a quadrant.

Cor. 2. If (Cor. 4. Prop. 4.), the angles at B and C each $= 90^\circ$, and the angle at A is nearly $180^\circ = 180^\circ - x$, x being a very small angle, then the side of the supplemental triangle opposite to A is equal to x; and the sum of the three sides of the supplemental triangle $=$ semicircle $+ x$, $= 180^\circ + x$.

Prop. IX. The sum of the three angles of a spherical triangle is $>$ 2 right angles $<$ 6 right angles.

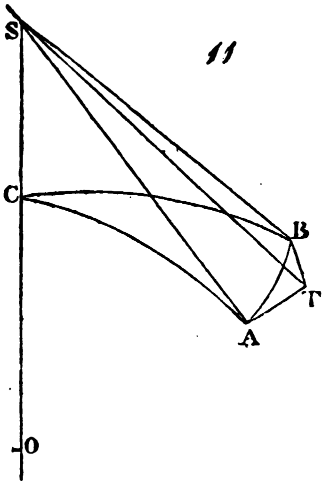
For, the angles of the triangle + sides of the supplemental triangle $= 180^\circ + 180^\circ + 180^\circ = 6 \times 90^\circ$; $\therefore$ since the sides of the supplemental triangle must have some magnitude, the angles of the triangle must be less than $6 \times 90^\circ$: again, by Prop. 3, p. 127, the sides of any triangle; and $\therefore$ the sides of the supplemental triangle, are less than $4 \times 90^\circ$, $= 4 \times 90 - x$; suppose, consequently, the angles of the triangle $= 6 \times 90^\circ - (4.90^\circ - x) = 2 \times 90^\circ + x$.

Cor. 1. Hence a spherical triangle may have two or three right angles, two or three obtuse angles.

Cor. 2. If the angles at B and C are right angles, AB, AC are quadrants, and A is the pole of BC: if also the angle at A is a right angle, the triangle ABC coincides with the supplemental or polar triangle, and the triangle ABC is contained eight times in the surface of the sphere.

Prop. X. The angles at the base of an isosceles spherical triangle are equal.

In the triangle ACB, let $AC = BC$, draw the tangents AS, BS, which are equal and which cut their secant OS in the common point S. Draw also from A and B two tangents AT, BT, which, by *Euclid*, Book III. Prop. 37. are equal.

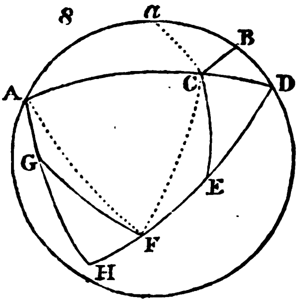


Hence, in the triangles SAT, SBT; SA, AT, ST, are respectively equal to SB, BT, ST; $\therefore$ by *Euclid*, Book I. Prop. 8, the angle SAT = the angle SBT, and $\therefore$ by Prop. 6, page 129, the spherical angle at A = the spherical angle at B.

To prove the reverse proposition, that is, to prove if the angles are equal, the sides are equal; take the supplemental triangle; then, since its sides opposite to the angles at A and B are supplemental of equal angles, the two sides are equal, and the supplemental triangle is isosceles; $\therefore$ by this proposition, the angles at the base are equal; $\therefore$ their supplements which are the sides AC, BC, are also equal.

Prop. XI. In a spherical triangle the greater side is opposite to the greater angle.

In the triangle ACB, let $\angle\, ACB$ be greater than ABC;



make $BCa = CBa$; $\therefore$ $Ca = aB$ (Prop. 10.); but $Aa + aC$ is greater than AC; $\therefore$ $Aa + aB$, or AB is greater than AC, by Prop. 2, page 127.

Prop. XII. The surface of the sphere included between the arcs DN, DM, is proportional to the angle NDM or the arc NM. See Fig. 9.

For, if the circumference $ANMB$ be divided into equal parts as NM, and great circles be drawn from D through the points of division as N, M, the portions of the surface, such as NDM, will be all similar and equal; hence, if AM contains NM, p times, or, if $AM = p \times NM$, the surface ADM will $= p \times NDM$.

Cor. When DM coincides with DB, the angle ADB and its measure $AMB = 180^\circ$: hence, if $S =$ the whole surface of a sphere, and if $a =$ the angle NDM, or the arc NM, the surface

$NDM = \frac{S}{4} \times \frac{a}{180}$; or, since S = area of 4 great circles of a sphere, (See Simpson's *Fluxions*, page 189, vol. I. or Vince's, page 89,) $= 4\pi$ ($\pi = 3.14159$, &c.) the surface $NDM = \pi\,\frac{a}{180}$.

The surface may be differently expressed, thus: π is the circumference of a circle the radius of which is $\frac{1}{2}$; $\therefore 2\pi$ represents the circumference of a circle, such as $ANMB$, the radius of which is supposed $= 1$, and as we represent arcs of great circles by the number of degrees which they contain, $2\pi = 360^{\circ}$, and $\frac{S}{4} = \pi = 180^{\circ}$: hence, the surface $NDM = 180^{\circ}.\frac{a}{180} = a^{\circ}$.

Prop. XIII. The measure of the surface of a spherical triangle is the difference between the sum of its three angles and two right angles.

Let the triangle be ABC, a, b, c, representing the magnitudes of the angles at A, B, C; let P = surface $BCmB$, $Q = mCnm$, $R = ACnA$; produce the arcs Cm, Cn till they meet at e

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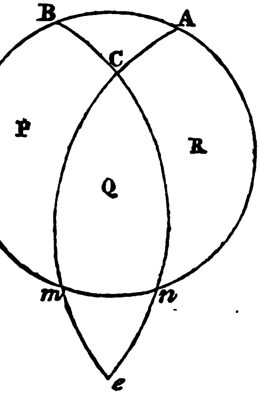


(which will be on the hemisphere opposite to that represented by $ABmnA$) then, each of the angles at C and e, equals the angle of the planes in which the arcs Cme, Cne, lie; $\therefore$ the angles at C and e are equal.

Again, the semicircles ACm, Cme; BCn, Cne are equal; or, $AC + Cm = Cm + me$, and $\therefore AC = me$, and $BC = ne$: and the triangle men = the triangle ABC; let x = its area, then, by Cor. to Prop. 12.,

$$x + P = \frac{S}{2} \cdot \frac{a}{180}$$

$$x + Q = \frac{S}{2} \cdot \frac{c}{180}, \text{ and, } x + P + Q + R = \frac{S}{2}$$

$$x + R = \frac{S}{2} \cdot \frac{b}{180};$$

consequently, by addition,

$$2x + (x + P + Q + R) \text{ or } 2x + \frac{S}{2} = \frac{S}{2} \cdot \left(\frac{a + b + c}{180}\right);$$

$$\therefore x = \frac{S}{4 \cdot (180)}(a + b + c) - \frac{S}{4} = a + b + c - 180^{\circ},$$

by the Cor^{y}. to the last Proposition.

This result is the same as that given by Caswell, in Wallis's Works, Vol. II. page 875, who attributes the Theorem to Albert Girard *.

Cor. 1. Since by Prob. 9, page 121, the limits of $a+b+c$ are 180° and 540°, the area of the triangle ABC may be equal to any number of degrees between 0 and 360°.

Cor. 2. If each of the angles a, b, c, $= 90^{\circ}$, the area of the triangle $ABC = 270^{\circ} - 180^{\circ} = 90^{\circ}$, which is $\frac{1}{8}$th of the whole surface, since $4\pi = 8\ (90^{\circ})$: this agrees with Cor. 2. Prop. 9.

---

* This expression for the value of the area was a merely speculative truth, and continued barren for more than 150 years, till 1787, when General Roy employed it, in correcting the spherical angles of observation made in the great Trigonometrical Survey, *Phil. Trans.* vol. VIII, year 1790, page 163. See also *Mem. Acad.* Paris, 1787, page 358, and *Mem. Inst.* vol. VI, p. 511.

Cor. 3. The area of *DMN* (see Fig. 9), $= \frac{S}{4} \times \frac{e}{180}$ (if e be its angle), $= e$; if we conceive *DN*, *DM*, to be continued till they meet in the opposite pole, and name the space included between the great circles a *Lune*, then the lune $= 2 \,.\, DNM = 2e$: but the area of $ABC = a + b + c - 180^\circ$: equate this with the area of the lune, and

$$e = \frac{a+b+c}{2} - 90^\circ;$$

which is the value of the angle *NDM*, when the area of the lune equals the area of the triangle.

Prop. XIV. If n be the number of the sides of a spherical polygon, its surface equals the sum of its angles, minus the product of two right angles multiplied by $n-2$.

Let the polygon be *AGFEÇA*: divide it by the means of the arcs of great circles into triangles, then

$$\begin{aligned} \text{area } AGF &= \angle AGF + \angle GFA + \angle GAF - 180^\circ \\ \text{area } AFC &= \angle FAC + \angle AFC + \angle ACF - 180^\circ \\ \text{area } CFE &= \angle CEF + \angle CFE + \angle FCE - 180^\circ \end{aligned}$$

$\therefore$ area of polygon $= \angle AGF + (\angle FAC + \angle GAF) + (\angle GFA + \angle AFC + \angle CFE) + \angle CEF + (\angle FCE + \angle ACF) - 3 \times 180^\circ$

$= \angle AGF + \angle GAC + \angle GFE + \angle CEF + \angle ECA - (5 - 2)\, 180^\circ$.

This demonstration proves the Proposition to be true for a polygon of five sides, and a similar demonstration will prove it true for a polygon of n sides: for, it is plain, if, instead of *AC*, we introduce *Ca*, *Aa*, that is, if we introduce an additional side, we introduce an additional triangle, and consequently we must introduce an additional 180° to be subtracted, that is, the negative part will become $-\{(5-2)\,180^\circ + 180^\circ\}$ or $-(6-2)\,180^\circ$.

The preceding Propositions belong, more properly, to Spherical Geometry than to Spherical Trigonometry; they have, however, been here inserted, because they exhibit certain properties of spherical triangles rather curious, and very easy of demonstration; so that, if not of essential use in assigning the relations between the angles and sides of spherical triangles, (which it is the special object of Trigonometry to assign,) they will not materially divert or impair the Student's attention.

In the next Chapter, we will proceed to deduce those formulæ by which, the relations between the sides and the angles of spherical triangles are expressed.

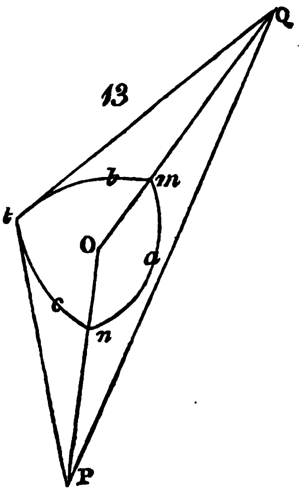
CHAP. IX.

On the Expressions for the Cosine and Sine of the Angle of a Spherical Triangle in Terms of the Sines and Cosines of the Sides.

WE will begin this Chapter with establishing in the following Problem, the fundamental formula, from which all the methods and forms of solution will be deduced: it corresponds to the fundamental one of Plane Trigonometry, inserted in page 24; and the Student who understands, in principle, the use made of that latter formula, possesses, in fact, the clue to the subsequent demonstrations of Spherical Trigonometry.

PROP. XV. PROBLEM.

It is required to express the cosine of the angle of a spherical triangle in terms of the sines and cosines of the sides.



Let the triangle be $m t n$, let the sides be a, b, c, and the opposite angles A, B, C; conceive O to be the centre of the sphere, and draw the two tangents tQ, tP, at the point t, to the arcs tm, tn; then $\angle QtP = \angle A$: by Prop. 6; and the angle at O is measured by mn, or a; and, by the definition of the secant,

$$OQ \text{ is sec. } b$$
$$OP \text{ is sec. } c.$$

The principle of the demonstration that follows, is, to obtain two values of QP, one from the triangle POQ, the other from PtQ, and then to compare them: now by *Euclid*, Prop. 13, Book II, or Prob. 1 of this Treatise:

in ΔPOQ, $PQ^2 = \text{sec.}^2 b + \text{sec.}^2 c - 2 \text{ sec. } b \,.\, \text{sec. } c \text{ cos. } a$ [rad. 1]
in ΔPtQ, $PQ^2 = \text{tan.}^2 b + \text{tan.}^2 c - 2 \text{tan. } b \,.\, \text{tan. } c \,.\, \text{cos. } A$.

Subtract the lower expression from the upper,

then, since $\text{sec.}^2 b - \text{tan.}^2 b = (\text{rad.})^2 = 1$, we have

$$0 = 1 + 1 + 2 \text{ tan. } b \,.\, \text{tan. } c \,.\, \text{cos. } A - 2 \text{ sec. } b \,.\, \text{sec. } c \,.\, \text{cos. } a$$

and thence,

$$\text{cos. } A = \frac{\text{cos. } a \,.\, \text{sec. } b \,.\, \text{sec. } c - 1}{\text{tan. } b \,.\, \text{tan. } c} = \frac{\text{cos. } a - \text{cos. } b \,.\, \text{cos. } c}{\text{cos. } b \text{ cos. } c \,.\, \text{tan. } b \,.\, \text{tan. } c}$$

$$= [\text{since cos. } b \,.\, \text{tan. } b = \text{sin. } b] \; \frac{\text{cos. } a - \text{cos. } b \,.\, \text{cos. } c}{\text{sin. } b \,.\, \text{sin. } c} \ldots\ldots\ldots [a]$$

and similarly, since the process for finding cos. B, cos. C, will be exactly the same, changing a for b, b for a, &c. the result must be similar, that is,

$$\text{cos. } B = \frac{\text{cos. } b - \text{cos. } a \,.\, \text{cos. } c}{\text{sin. } a \,.\, \text{sin. } c} \ldots\ldots\ldots\ldots [b]$$

$$\text{cos. } C = \frac{\text{cos. } c - \text{cos. } a \,.\, \text{cos. } b}{\text{sin. } a \,.\, \text{sin. } b} \ldots\ldots\ldots\ldots [c].$$

Cor. 1. Hence, since

$$\text{cos. } c = \text{sin. } a \,.\, \text{sin. } b \,.\, \text{cos. } C + \text{cos. } a \,.\, \text{cos. } b$$

by substituting this value of cos. c in the expression [a], we have

$$\text{cos. } A = \frac{\text{cos. } a - \text{cos. } b \,.\, \text{sin. } b \,.\, \text{sin. } a \,.\, \text{cos } C - \text{cos. } a \text{ cos.}^2 b}{\text{sin. } b \,.\, \text{sin. } c}.$$

But, $\text{cos. } a - \text{cos. } a \,.\, \text{cos.}^2 b = \text{cos. } a\,(1 - \text{cos.}^2 b) = \text{cos. } a \,.\, \text{sin.}^2 b$;

$$\therefore \cos. A = \frac{\cos. a \sin.^2 b - \cos. b . \sin. b . \cos. C . \sin. a}{\sin. b . \sin. c}$$

$$= \frac{\cos. a . \sin. b - \cos. b . \sin. a . \cos. C}{\sin. c} :$$

Similarly, $\cos. B = \dfrac{\cos. b . \sin. a - \cos. a . \sin. b . \cos. C}{\sin. c}$;

$$\therefore \cos. A + \cos. B = \frac{(\cos. a . \sin. b + \cos. b . \sin. a)(1 - \cos. C)}{\sin. c}$$

$$= \frac{\sin. (a + b) \, 2 \sin.^2 \dfrac{C}{2}}{\sin. c} .$$

PROP. XVI. PROBLEM.

It is required to express the sine of the angle of a spherical triangle in terms dependent on its sides.

By the last Proposition, $\cos. A = \dfrac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c}$;

$$\therefore 1 + \cos. A =$$

$$\frac{\cos. a - (\cos. b . \cos. c - \sin. b . \sin. c)}{\sin. b . \sin. c} = \frac{\cos. a - \cos. (b + c)}{\sin. b . \cos. c}$$

by the form [2], p. 130; but by the form [8], page 33,

$$\cos. a - \cos. (b + c) = 2 \sin. \left(\frac{a + b + c}{2}\right) \sin. \left(\frac{b + c - a}{2}\right) .$$

Hence,

$$1 + \cos. A = \frac{2}{\sin. b \sin. c} \times \sin. \left(\frac{a + b + c}{2}\right) \sin. \left(\frac{b + c - a}{2}\right)$$

$$\text{or, } = \frac{2}{\sin. b \sin. c} \times \sin. \left(\frac{a + b + c}{2}\right) . \sin. \left(\frac{a + b + c}{2} - a\right)$$

Similarly,

$$1 - \cos. A = \frac{\cos. b . \cos. c + \sin. b \sin. c - \cos. a}{\sin. b . \sin. c}$$

$$= \frac{\cos.(b-c) - \cos. a}{\sin. b . \sin. c} \text{ (by the form [4], p. 31.)}$$

$$= \frac{2}{\sin. b \sin. c} \times \sin.\left(\frac{a+b-c}{2}\right)\sin.\left(\frac{a+c-b}{2}\right) \text{ by form [8], p. 33;}$$

$$\text{or, } = \frac{2}{\sin. b . \sin. c} \times \sin.\left(\frac{a+b+c}{2} - b\right) \sin.\left(\frac{a+b+c}{2} - c\right).$$

Hence, if we multiply together $1 + \cos. A$, and $1 - \cos. A$, the product of which is $\sin.^2 A$, and substitute S instead of $\frac{a+b+c}{2}$, we shall have $\sin.^2 A =$

$$\frac{4}{\sin.^2 b . \sin.^2 c} \times \sin. S . \sin.(S-a) \sin.(S-b) \sin.(S-c),$$

and consequently,

$$\sin. A =$$

$$\frac{2}{\sin. b . \sin. c} \sqrt{\{ \sin. S . \sin.(S-a) \sin.(S-b) \sin.(S-c) \}}\,*.$$

Cor. 1. If we wish to compute $\sin. B$, we must begin from $\cos. B = \frac{\cos. b - \cos. a . \cos. c}{\sin. a . \sin. c}$, and proceed exactly in the steps of the former process: the result will be a fraction, the numerator of which is the numerator of the above fraction for the $\sin. A$, and the denominator will be $\sin. a \sin. c$; call the common numerator N, then

$$\sin. A = \frac{N}{\sin. b . \sin. c}, \sin. B = \frac{N}{\sin. a . \sin. c}, \sin. C = \frac{N}{\sin. a . \sin. b}.$$

Cor. 2. Hence $\frac{\sin. A}{\sin. B} = \frac{N}{\sin. b . \sin. c} \times \frac{\sin. a . \sin. c}{N} =$

$\frac{\sin. a}{\sin. b}$, and $\frac{\sin. A}{\sin. C} = \frac{\sin. a}{\sin. c}$;

or (if these equations be expressed, after the manner of expressing a proposition) *The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.*

---

* The Student, for his own convenience, is desired to compare these expressions and the manner of deducing them, with the corresponding ones in Plane Trigonometry.

Right-angled spherical triangles may be considered as particular cases of oblique. The solutions of the latter, then, would necessarily include those of the former; and, accordingly, if we wished to generalise as much as we could generalise, it would not be requisite separately to consider them. Since, however, it is our object to render investigation as simple and as easy as it is possible to the Student, we shall not avail ourselves of this abridgment, and seek to be compendious, but proceed, in the ensuing Chapter, to treat distinctly of the cases of right-angled spherical triangles.

CHAP. X.

Formulæ of Solution for Right-angled Spherical Triangles.—Affections of Sides and Angles.—Circular Parts.—Naper's Rules.—Quadrantal Triangles.—Examples.

PROP. XVII. PROBLEM.

IT is required to investigate formulæ of solution for all the cases of right-angled spherical triangles *.

These are to be derived from the fundamental expressions [*a*], [*b*], [*c*], given in page 139: let C be the right angle and $= 90^0$; $\therefore \cos. C = 0$;

hence,

$$\cos. C, \text{ or } 0 = \frac{\cos. c - \cos. a \cdot \cos. b}{\sin. a \cdot \sin. b}, \text{ and } \therefore \cos. c = \cos. a \cdot \cos. b \quad [1]$$

hence, $\cos. a = \frac{\cos. c}{\cos. b}$, $\cos. b = \frac{\cos. c}{\cos. a}$; substitute these values respectively, for $\cos. a$ and $\cos. b$, in the expressions for $\cos. A$, $\cos. B$, and we have

$$\cos. A = \frac{\cos. a - \cos. b \cdot \cos. c}{\sin. b \cdot \sin. c} =$$

$$\frac{1}{\sin. b \cdot \sin. c} \left(\frac{\cos. c}{\cos. b} - \cos. b \cdot \cos. c \right) =$$

$$\frac{\cos. c\,(1 - \cos.^2 b)}{\cos. b \cdot \sin. b \cdot \sin. c} = \frac{\cos. c \cdot \sin.^2 b}{\cos. b \cdot \sin. b \cdot \sin. c} =$$

---

* Spherical triangles that have one right angle only, are the subject of investigation: and spherical triangles that have two right angles, and three right angles, are excluded.

$$\frac{\cos. c \cdot \sin. b}{\cos. b \cdot \sin. c} = \cot. c \,.\, \tan. b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [2]$$

Similarly,

$$\cos. B = \frac{\cos. c\,(1 - \cos.^2 a)}{\cos. a \,.\, \sin. a \,.\, \sin. c} = \frac{\cos. c \,.\, \sin. a}{\cos. a \,.\, \sin. c} = \cot. c \,.\, \tan. a \ldots . [3]$$

Now in [1] a, b, c are involved, and two of these being given the third may be found: in [2] A, c, b are involved: and if we choose, symbolically, to represent by (A, c, b) the form in which three quantities as A, c, b, are involved, then, similarly, the other several forms that can arise by combining angles and sides may be thus represented :

(c, a, b); $\{(A, c, b),\ (B, c, a)\}$; $\{(B, c, b)\ (A, c, a)\}$ $\{(A, a, b)\ (B, a, b)\}$; $\{(A, B, a)\ (A, B, b)\}$; (A, B, c): which are in number 10, as they ought to be; the combination in 5 things, 3 and 3 together, being $\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10$: those combinations that are similar, such as the second and third, are included within brackets.

The forms [1], [2], [3], have been already deduced, and the remaining ones must be deduced from them, and from the form $\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}$, by the common process of substitution and elimination,

(c, a, b), that is, $\cos. c = \cos. a \,.\, \cos. b \ldots\ldots\ldots\ldots [1]$
(A, c, b) $\cos. A = \cot. c \,.\, \tan. b \ldots\ldots\ldots\ldots [2]$
(B, c, a) $\cos. B = \cot. c \,.\, \tan. a \ldots\ldots\ldots\ldots [3]$

(B, c, b)

this is Cor. to Prop. 16, for $\frac{\sin. B}{\sin. C} = \frac{\sin. b}{\sin. c}$; $\therefore$ since $\sin. C = 1$,

$$\sin. B = \frac{\sin. b}{\sin. c}, \text{ or } \sin. b = \sin. c \cdot \sin. B \quad [4]$$

(A, c, a) is exactly similar;

$$\therefore \sin. B = \frac{\sin. a}{\sin. c}, \text{ or } \sin. a = \sin. c \cdot \sin. A \quad [5]$$

(A, a, b): $\cos. A = \cot. c . \tan. b$ by [2], but $\cot. c = \dfrac{\cos. c}{\sin. c}$,

and $\cos. c = \cos. a . \cos. b$ by [1], and $\sin. c = \dfrac{\sin. a}{\sin. A}$ by [5], p. 144,

$\therefore \cos. A = \tan. b . \dfrac{\cos. a . \cos. b . \sin. A}{\sin. a}$,

$\therefore \cot. A = \cot. a . \sin. b \ldots\ldots\ldots [6]$

(B, a, b) is similar, and $\therefore \cot. B = \cot. b . \sin. a \ldots\ldots\ldots [7]$

(A, B, a): divide [2] by [4]; then $\dfrac{\cos. A}{\sin. B} = \dfrac{\cos. c}{\cos. b} = \cos. a$

from [1] or, $\cos. A = \cos. a . \sin. B \ldots\ldots\ldots\ldots [8]$

(A, B, b) is similar, and $\cos. B = \cos. b . \sin. A \ldots\ldots\ldots\ldots [9]$

(A, B, c): multiply together [6] and [7]

and $\cot. A . \cot. B = \cos. b . \cos. a = \cos. c \ldots\ldots\ldots\ldots\ldots [10]$

Thus are all the forms easily deduced from [1], [2], [3]. Indeed, the quantities that are to be combined, indicate plainly the proper process of elimination; but these forms, although very simple and fitted for logarithmic computation, are not easily remembered, and therefore an artificial memory has been supplied to the Student and Computist, by rules, known by the title of *Naper's Rules for Circular Parts*; and in the whole compass of mathematical science there cannot be found, perhaps, rules which more completely attain that, which is the proper object of rules, namely, facility and brevity of computation.

The rules and their description are as follow:

Description of Circular Parts.

The right angle is not considered; the complements of the two other angles, the complement of the hypothenuse, the two sides, making in all five quantities, are called by Naper, *circular parts*. Any one of these circular parts may be called a *middle part* (M), and, then, the two circular parts immediately adjacent to the right and left of M are called *adjacent parts*; the other two remaining circular parts, each separated from M the middle part, by an adjacent part, are called *opposite parts*, or *opposite extremes*: thus, let the side a be M,

then, comp. B, or $90^\circ - B$, and b are adjacent parts,
and, comp. c, or $90^\circ - c$, and $90^\circ - A$, opposite parts.

If $90^\circ - A$ be M,

$90^\circ - c$ and b are adjacent parts; $90^\circ - B$ and a, opposite parts.

This necessary explanation being premised, we come to

Naper's Rules.

1. The rectangle of the sin. M and radius = rectangle of the tangents of adjacent parts.

2. The rectangle of the sin. M and radius = rectangle of the cosines of the opposite parts.

There is no separate and independent proof of these rules, but the rules will be manifestly just, if it can be shewn that they comprehend every one of the ten results, [1], [2], [3], &c. given, in pages 144, 145; for, those results solve every case in right-angled spherical triangles.

	M.	Adjacent Parts.	Opposite Parts.
1. $\cos. c = \cos. a . \cos. b$, or, $\sin. (90^\circ - c) = \cos. a . \cos. b$	$90^\circ - c$		a, b
2. $\cos. A = \cot. c . \tan. b$, or, $\sin. (90^\circ - A) = \tan. (90^\circ - c) \tan. b$	$90^\circ - A$	$90^\circ - c, b$	
3. $\cos. B = \cot. c . \tan. a$, or, $\sin. (90^\circ - B) = \tan. (90^\circ - c) \tan. a$	$90^\circ - B$	$90^\circ - c, a$	
4. $\sin. b = \sin. c . \sin. B$, or, $\sin. b = \cos. (90^\circ - c) \cos. (90^\circ - B)$	b		$90^\circ - c, 90^\circ - B$
5. $\sin. a = \sin. c . \sin. A$, or, $\sin. a = \cos. (90^\circ - c) \cos. (90^\circ - A)$	a		$90^\circ - c, 90^\circ - A$
6. $\cot. A = \cot. a . \sin. b$, or, $\sin. b = \tan. (90^\circ - A) . \tan. a$	b	$90^\circ - A, a$	
7. $\cot. B = \cot. b . \sin. a$, or, $\sin. a = \tan. (90^\circ - B) . \tan. b$	a	$90^\circ - B, b$	
8. $\cos. A = \cos. a . \sin. B$, or, $\sin. (90^\circ - A) = \cos. a . \cos. (90^\circ - B)$	$90^\circ - A$		$a, 90^\circ - B$
9. $\cos. B = \cos. b . \sin. A$, or, $\sin. (90^\circ - B) = \cos. b . \cos. (90^\circ - A)$	$90^\circ - B$		$b, 90^\circ - A$
10. $\cot. A . \cot. B = \cos. c$, or, $\sin. (90^\circ - c) = \tan. (90^\circ - A) \tan. (96^\circ - B)$	$90^\circ - c$	$90^\circ - A, 90^\circ - B$	

This is a complete proof of the truth of the rules, and, as we have already said, the only kind of proof which the rules admit of; but, after the proof, the rules ought to be used, and the formulæ having performed the service of proving them, are then superseded. The rules ought to be used also, not only in the immediate solution of right-angled spherical triangles, but in deducing, where they can be so made subservient, the properties of oblique triangles. We here allude to those properties announced in the Propositions 24, 25, 26, 27 of Robert Simson's *Trigonometry*. These immediately appear on the application of Naper's Rules, and their deduction is so obvious, that it is, practically, against the interest of the Student to make them the subject of three or four formal Propositions; since it is not worth the while to burthen the memory with the terms and enunciation of a Proposition, for the sake of formally making one or two steps in the process of deduction.

The forms for the solution of right-angled triangles have been deduced from the general expressions for the cos. A, cos. B, cos. C, in which, as in a particular case, $C = 90°$. The same general expressions may also be used for any other case, in which, to any one or more of the sides or angles particular values should be assigned: for instance, the side c may be a quadrant $= 90°$, in which case the triangle has been called a quadrantal triangle; in this case

$$\cos.\ C = \frac{\cos.\ c - \cos.\ a\,.\cos.\ b}{\sin.\ a\,.\sin.\ b} = -\frac{\cos.\ a\cdot\cos.\ b}{\sin.\ a\,.\sin.\ b}\ [\text{since } \cos.\ c = 0]$$

$$\text{and } \therefore \sin.\ (90° - C) = -\cot.\ a\,.\cot.\ b$$

$$= -\tan.\ (90° - a)\tan.\ (90° - b)\ \ldots\ldots\ldots [a]$$

$$\text{Again, } \cos.\ B = \frac{\cos.\ b - \cos.\ a\,.\cos.\ c}{\sin.\ a\,.\sin.\ c} = \frac{\cos.\ b}{\sin.\ a}$$

$$\cos.\ A = \frac{\cos.\ a - \cos.\ b\,.\cos.\ c}{\sin.\ b\,.\sin.\ c} = \frac{\cos.\ a}{\sin.\ b}.$$

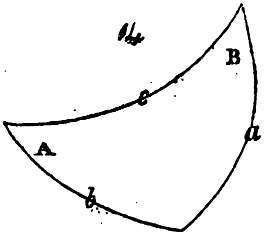
From these three equations, and the additional one, viz. $\frac{\sin.\ A}{\sin.\ B} = \frac{\sin.\ a}{\sin.\ b}$, may be deduced all the cases in quadrantal, as they have already been, in right-angled spherical triangles.

If, in the above expression [a], we call $90^\circ - C$ the middle part, and $90^\circ - a$, $90^\circ - b$ which are the complements of the sides, the adjacent parts, the above result might be comprehended under a Rule like Naper's; but, if a case like that occurs, the surest method, and, perhaps, on the whole, the most expeditious for the Student, and computist, would be to take the supplemental or polar triangle, and to solve it by Naper's Rule; for, the angle in the supplemental triangle opposite to $c = 180^\circ - c = 180^\circ - 90^\circ$, is a right angle: thus, in the case adduced, the angles of the supplemental triangle are $180^\circ - a$, $180^\circ - b$, and the hypothenuse is $180^\circ - C$; therefore by Naper's first rule, $1 \times \sin.(C - 90^\circ) = \tan.(a - 90^\circ) . \tan.(b - 90^\circ)$ the same result as [a].

Examples of the solution of right-angled spherical triangles.

Example 1.

$$\text{Given} \begin{cases} A = 23^\circ\ 27'\ 42'' \\ b = 10\ \ 39\ \ 40 \end{cases} \qquad \text{Required} \begin{cases} c \\ a \\ B \end{cases}$$



c determined.

$90^\circ - A$ middle part, $90^\circ - c$, b, adjacent parts;

$\therefore r . \cos. A = \cot. c . \tan. b$..............[1st rule]

$\therefore \log. \cot. c = \log. r + \log. \cos. A - \log. \tan. b$

$\log. r = 10$;

$\therefore \log. r + \log. \cos. 23^\circ\ 27'\ 42''$........................ $= 19.9625240$

$\log. \tan. 10\ \ 39\ \ 40$........................ $= \ \ 9.2747329$

$\log. \cot. c$........................ $= 10.6877911$

$\therefore c = 11^\circ\ 35'\ 49''$.

a determined.

∴ b middle part, $90^\circ - A$, a, adjacent parts;

∴ $r \,.\, \sin.\, b = \cot.\, A \,.\, \tan.\, a$ [1st rule, p. 146,]

∴ $\log.\, \tan.\, a = \log.\, r + \log.\, \sin.\, b - \log.\, \cot.\, A$

$10 + \log.\, \sin.\, 10^\circ\ 39'\ 40''$ .... $= 19.2671709$
$\log.\, \cot.\, 23\ \ 27\ \ 42$ .... $= 10.3624932$

∴ $\log.\, \tan.\, a$ .... $= 8.9046777$

∴ $a = 4^\circ\ 35'\ 26''$.

B determined.

$90^\circ - B = M$, $90^\circ - A$, b opposite parts;

∴ $r \,.\, \cos.\, B = \sin.\, A \,.\, \cos.\, b$ ....[2d rule]

∴ $\log.\, \cos.\, B + \log.\, r = \log.\, \sin.\, A + \log.\, \cos.\, b$

$\log.\, \sin.\, 23^\circ\ 27'\ 42''$ .... $= 9.6000308$
$\log.\, \cos.\, 10\ \ 39\ \ 40$ .... $= 9.9924380$

∴ $\log.\, \cos.\, B + 10$ .... $= 19.5924688$

∴ $B = 66^\circ\ 58'\ 1''$.

The above is the solution of a Problem in Astronomy, in which from the obliquity A, and the right ascension b, as given quantities, the longitude c of the Sun, and the declination a are required.

Example 2.

$$\text{Given} \left\{ \begin{array}{l} a = 27^\circ . 48' \\ c = 71 \,.\, 39.37 \end{array} \right\} \text{Required} \left\{ \begin{array}{c} b \\ B \end{array} \right\}$$

b determined.

$90 - c = M$, a, b, opposite parts.

∴ $r \,.\, \cos.\, c = \cos.\, a \,.\, \cos.\, b$ ....2d rule;

∴ $\log.\, \cos.\, b = 10 + \log.\, \cos.\, c - \log.\, \cos.\, a$,

$10 + \log.\, \cos.\, 71^\circ\ 39'\ 37''$ .... $= 19.4978286$
$\log.\, \cos.\, 27\ \ 48\ \ \ 0$ 9.9467376

∴ $\log.\, \cos.\, b$ .... $= 9.5510910$

and $b = 69^\circ\ 9'\ 48''$.

B determined.

$90 - B = M$, $90 - c$, a, adjacent parts;

$\therefore\ r.\cos. B = \cot. c . \tan. a$ .... by 1st rule

$\log.\cos. B + 10 = \log.\cot. c + \log.\tan. a$,

$\log.\cot. 71^\circ\, 39'\, 37''$ .... $= 9.5204674$

$\log.\tan. 27\ \ 48\ \ 0$ .... $= 9.7220085$

$10 + \log.\cos. B$ .... $= 19.2424759$

$\therefore\ B = 79^\circ\, 56'\, 4''$.

This is the solution of an Astronomical Problem, in which, from the latitude of a place $= 90^\circ - 27^\circ\, 48' = 62^\circ\, 12'$, and the latitude of the Sun at six o'clock $= 90^\circ - 71^\circ\, 39'\, 37'' = 18^\circ\, 20'\, 23''$, it is required to find the Sun's declination, which, by the result, would be $90^\circ - 69^\circ\, 9'\, 48'' = 20^\circ\, 50'\, 12''$; the angle $B = 79^\circ\, 56'\, 11''$, in the same Problem, is the Sun's azimuth.

Example 3.

C not a right angle, but *c*, the side opposite to it, $= 90^\circ$.

$$\text{Given}\ \left\{\begin{array}{l} a = 32^\circ\, 57'\, 6'' \\ b = 66\ \ 32\ \ 0 \end{array}\right\}\ \text{Required}\ \left\{\begin{array}{l} B \\ C \end{array}\right.$$

C determined.

By the expression [*a*], p. 139,

$$\cos. C = \frac{\cos. c - \cos. a . \cos. b}{\sin. a . \sin. b}$$

$$= -\frac{\cos. a . \cos. b}{\sin. a . \sin. b},\ \text{since}\ \cos. c = 0$$

$$= -\cot. a . \cot. b;$$

$\therefore$ in logarithms,

$$\log.\cos. C = \log.\cot. a + \log.\cot. b - 10.$$

Now $- 10 + \log.\cot. 32^\circ\, 57'\, 6''$ .... $= .1882850$

$\log.\cot. 66\ \ 32\ \ 0$ .... $= 9.6376106$

$\therefore\ \cos. C$ .... $= 9.8258956$

$\therefore\ C = 180^\circ - (47^\circ\, 57'\, 16'') = 132^\circ\, 2'\, 44''$.

The supplement of $47^\circ\, 57'\, 16''$ is taken, since from the expression

$$\cos. C = -\cot. a . \cot. b$$

$\cos. C$ is negative, $\therefore\ C > 90^\circ$.

B determined.

By the expression [*b*], page 139,

$$\cos.\ B = \frac{\cos.\ b - \cos.\ a\,.\cos.\ c}{\sin.\ a\,.\sin.\ c}$$

$$= \frac{\cos.\ b}{\sin.\ a},\ \text{since } \cos.\ c = 0;$$

$$\therefore\ \log.\ \cos.\ B = 10 + \log.\ \cos.\ b - \log.\ \sin.\ a$$

$10 + \log.\ \cos.\ 66^\circ\ 32'\ 0''$ .... = 19.6001181

$\log.\ \sin.\ 32\ \ 57\ \ 6$ .... = 9.7355441

$\therefore\ \log.\ \cos.\ B$ .... = 9.8645740

$$\therefore\ B = 42^\circ\ 56'\ 12''.$$

Here, the Problem is most simply resolved from the fundamental expressions for the cosines of the angles: and there is no need that we recur to the supplemental triangle: if however we do, then since its two angles are $180^\circ - a$, $180^\circ - b$, and the adjacent side $= 180^\circ - C$, by Naper's first Rule, $r \sin.\ (C - 90^\circ) = \tan.\ (a - 90^\circ)\ \tan.\ (b - 90^\circ)$ or $-\ r\,.\cos.\ C = \cot.\ a\,.\cot.\ b$, as before; and similarly for *B*.

The above is the solution of an Astronomical Problem, in which, from the latitude $= 90^\circ - 32^\circ\ 57'\ 6'' = 57^\circ\ 2'\ 54''$, and the Sun's declination $= 90^\circ - 66^\circ\ 32' = 23^\circ\ 28'$, it is required to find the time at which the Sun rises.

In the solution of plane triangles, one of the cases, see p. 80, is ambiguous: in spherical right-angled triangles, there are three ambiguous cases: and these are when the quantities, *a*, *c*, *A* are to be found from *b* and *B* given: now, by Naper's Rules, *a*, *c*, *A* are given by these three equations;

$$r\,.\sin.\ a = \tan.\ b\,.\cot.\ B.$$
$$r\,.\sin.\ b = \sin.\ c\,.\sin.\ B;$$
$$r\,.\cos.\ B = \sin.\ A\,.\cos.\ b;$$

but after sin. *a*, sin. *c*, sin. *A*, are deduced from these equations, there is nothing to determine us, whether we ought to take *a*, *c*, *A*, or $180^\circ - a$, $180^\circ - c$, $180^\circ - A$, ought to be taken; for, the sines of the 3 latter quantities are the same as the sines of the 3 former: and it is easily shewn, that there are two right-angled spherical triangles, which have an angle and side opposite the same in both, but in which the remaining sides, and the remaining angle of the one, are respectively the supplements of the remaining sides and the remaining angle of the other.

The other cases are not ambiguous, and Naper's Rules, with an attention to the signs of the quantities involved, will enable us to remove the ambiguity which some of these cases appear to have: thus, if b be the middle part, $90^\circ - c$, and $90^\circ - B$, the opposite parts, then $\sin. b = \sin. c . \sin. B$: if b be required from this equation, will it not be doubtful, whether b or $180^\circ - b$, [since $\sin. b = \sin. (180^\circ - b)$] ought to be taken? The ambiguity is removed by this property, that, if B be $>$ or $< 90^\circ$, b is $>$ or $< 90^\circ$: for by Naper's 1st Rule, $\sin. a = \cot. B . \tan. b$; now, $\sin. a$ is positive when a is between 0 and 180°, and if B be $> 90^\circ$, $\cot. B$ is negative; and consequently, $\tan. b = \frac{\sin. a}{\cot. B}$, is negative, and $b > 90^\circ$. If $B < 90^\circ$, $\cot. B$ is positive; $\therefore \tan. b. = \frac{\sin. a}{\cot. B}$ is positive: and $b < 90^\circ$; similarly, make $90^\circ - B = M$, and $90^\circ - c$, and a adjacent parts, then, $\cos. B = \cot. c . \tan. a$; $\therefore$ if c be sought, $\tan. c = \frac{\tan. a}{\cos. B}$; if a be $> 90^\circ$, and $B > 90^\circ$, then $\tan. a$ and $\cos. B$ are both negative; $\therefore \tan. c$ is positive, and consequently, c is $< 90^\circ$: if $a > 90^\circ$, and $B < 90^\circ$, or, if $a < 90^\circ$, and $B > 90^\circ$, then $\frac{\tan. a}{\cos. B}$ is negative; $\therefore \tan. c$ is negative; $\therefore c > 90^\circ$.

These considerations are so simple and so easily made, that it is, perhaps, better to let the Student endeavour to avail himself of similar ones, than to burthen his memory with the terms and results of formal Propositions: for it must be noticed, that in order to prevent the ambiguity of solutions in right-angled triangles, terms have been invented and propositions framed relative to the *affections* of the sides and angles: sides and their opposite angles being said to have the *same affection*, when each is less or greater than 90°: See Simson's *Euclid*, Prop. 13, p. 500, 8vo. edit. 1781.

It has been already proved, that sides and the opposite angles are each greater or less than 90°, that is, have the same affection: again, since by the form [1], p. 143,

$$\cos. c = \cos. a . \cos. b,$$

if a and b be both $> 90^\circ$, $\cos. a$ and $\cos. b$ are both negative; $\therefore$ their product, which $= \cos. c$, is positive, and $c < 90^\circ$; $\therefore$ if both be $< 90^\circ$, $\cos. c$ is positive, and $c < 90^\circ$: if a be $> 90^\circ$, and $b < 90^\circ$, or if $a < 90^\circ$, and $b > 90^\circ$, the product $\cos. a \cdot \cos. b$ is negative; $\therefore \cos. c$ is negative; $\therefore c$ is $> 90^\circ$. This may be easily translated into the terms and language which Robert Simson uses in his Trigonometry. See Prop. 14, *Spherical Trigonometry* at the end of Euclid's *Elements*, p. 500.

The several cases of right-angled spherical triangles being now solved, we will proceed in the next Chapter to the solution of oblique-angled triangles.

CHAP. XI.

Equations exhibiting the Relations of the Sides and Angles of Oblique-angled Spherical Triangles.—Formulæ of Solution deduced from such Equations. — Examples, &c.

In the cases of oblique-angled spherical triangles six quantities are concerned, a, b, c, A, B, C: and the general problem requires us to determine three of the six by means of the three others. We must have equations then between four of these quantities combined all possible ways; but the number of the combinations of six quantities, taken four and four, equals $\frac{6.5.4.3}{1.2.3.4}$, or 15. These combinations are

$$(a\,b\,c\,A),\ (a\,b\,c\,B),\ (a\,b\,c\,C)$$
$$(ABCa),\ (ABCb),\ (ABCc)$$
$$(a\,C\,b\,A),\ (a\,C\,b\,B),\ (a\,B\,c\,A),\ (a\,Bc\,C),\ (b\,A\,c\,B),\ (b\,A\,c\,C)$$
$$(a\,A\,b\,B),\ (a\,C\,c\,A),\ (b\,C\,c\,B).$$

Now, the number of combinations essentially different is the number of the horizontal rows, or four: for instance, the combinations of the first row depend on three similar equations:

$$\cos. A = \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c} \dots\dots\dots\dots [1]$$

$$\cos. B = \frac{\cos. b - \cos. a . \cos. c}{\sin. a . \sin. c} \dots\dots\dots\dots [2]$$

$$\cos. C = \frac{\cos. c - \cos. a . \cos. b}{\sin. a . \sin. b} \dots\dots\dots\dots [3].$$

The combinations of the fourth row depend on three similar equations,

$$\frac{\sin. a}{\sin. b} = \frac{\sin. A}{\sin. B};\ \frac{\sin. a}{\sin. c} = \frac{\sin. A}{\sin. C};\ \frac{\sin. b}{\sin. c} = \frac{\sin. B}{\sin. C},$$

and similarly for the remaining two rows: hence the solution of all the cases of oblique-angled triangles is reduced, in fact, to four equations, and these four equations must be deduced, as the equations in p. 144 were, by the ordinary processes of substitution and elimination.

We will now proceed to deduce these four equations.

First equation belonging to the form ($a\,b\,c\,A$),

$$\cos. A = \frac{\cos. a - \cos. b\,.\cos. c}{\sin. b\,.\sin. c}.$$

Second equation belonging to ($A B C\,a$).

In order to obtain this, eliminate cos. b, cos. c, sin. b, sin. c from the equations [1], [2], [3]: or, more simply to obtain it, take the supplemental triangle; then, if a', b', c' are its sides, A', B', C', its angles, we have, by the form [1],

$$\cos. A' = \frac{\cos. a' - \cos. b' \cos. c'}{\sin. b'\,.\sin. c'};$$

$$\begin{aligned} \text{but}\quad \cos. A' &= \cos. (180^\circ - a) = -\cos. a \\ \cos. a' &= \cos. (180^\circ - A) = -\cos. A \\ \cos. b' &= \cos. (180^\circ - B) = -\cos. B \\ \cos. c' &= \cos. (180^\circ - C) = -\cos. C \\ \sin. b' &= \sin. (180^\circ - B) = \sin. B \\ \sin. c' &= \sin. (180^\circ - C) = \sin. C; \end{aligned}$$

$$\text{consequently,}\quad -\cos. a = \frac{-\cos. A - \cos. B\,.\cos. C}{\sin. B\,.\sin. C};$$

$$\text{or,}\quad \cos. a = \frac{\cos. A + \cos. B\,.\cos. C}{\sin. B\,.\sin. C}.$$

Third equation belonging to ($a\,C\,b\,A$).

In order to obtain this, substitute in the equation [1], instead of cos. c, its value derived from the equation [3]: and instead of sin. c, substitute $\frac{\sin. C}{\sin. A}$ sin. a,

$$\text{then,}\quad \cos. A\,.\sin. b\,.\frac{\sin. C}{\sin. A}\,.\sin. a =$$

$$\cos. a - \cos. b (\cos. C . \sin. a . \sin. b + \cos. a . \cos. b)$$
$$= \cos. a \sin.^2 b - \cos. C . \sin. a \sin. b . \cos. b.$$

Hence, on dividing each side of the equation by $\sin. a \times \sin. b$, there results

$$\cot. A . \sin. C = \cot. a . \sin. b - \cos. C . \cos. b.$$

Fourth equation belonging to ($a\ A\ b\ B$).

This equation is deduced in Cor. 2. to Prop. 16, where it is proved, that $\frac{\sin. a}{\sin. b} = \frac{\sin. A}{\sin. B}$.

These four equations analytically resolve the Problem; or, by means of them, any three quantities being given, the fourth may be found; but it is plain from their inspection, that they do not afford convenient solutions, since none of them are under a form adapted to logarithmic computation; and even, if, in order to find one of the quantities involved in the equation, we were to express the equation under a form adapted to a logarithmic computation, such modified form would be useless, except in the case for which it was contrived: that is, would be useless, if one of the quantities, by which the required quantity was expressed, should itself be required to be computed, the previously required quantity becoming, in this second case, one of the given quantities: for instance, in the combination ($a b c A$) if A be the quantity sought, then, by p. 140,

$$1 + \cos. A, \text{ or, } 2 \cos.^2 \frac{A}{2} =$$

$$\frac{2}{\sin. b . \sin. c} \left\{ \sin. \left(\frac{a+b+c}{2}\right) \sin. \left(\frac{a+b+c}{2} - a\right) \right\}$$

whence, by a logarithmic computation, cos. A and A can be found; but, from such form, if A were given, and a required, a cannot be immediately and conveniently found: and, on this account, something more is required of the analyst, than mere equations that exhibit the possibility of solutions: he ought to furnish formulæ, from which, the quantity, whatever it be, side or angle, may by a direct, certain, and commodious process, be found. Formulæ then, as it will easily be seen, by which one quantity may be computed from others that are given, must be more in number than the equations which merely exhibit the relations of the quantities. This does not take place in right-angled spherical triangles; in which the formulæ of solution need not exceed in number the equations: for instance, the equation $\cos. c = \cos. a . \cos. b$ represents the relation between c, a, b; and from the same equation may be found, by a like logarithmic computation, either c from a and b; or a from b and c; or b from a and c; the same may be said of the second equation, that is, of $\cos. A = \cot. c . \tan. b$: either A, or c, or b, may, from the equation as it stands, be found with equal facility.

The solution of oblique-angled spherical triangles will be found, by what follows, to require six cases; and in using the foregoing forms of combination such as $(a\,b\,c\,A)$, the quantity sought will be placed last.

Cases of Oblique-angled Spherical Triangles.

PROP. 18. CASE 1st, $(a\,b\,c\,A)$.

The sides a, b, c, are given, and the angle A is required.

First Method of Solution.

By Prob. 16. page 140, if $S = \frac{a+b+c}{2}$,

$$\sin. A = 2 . \frac{\surd \{ \sin. S . \sin. (S-a) . \sin. (S-b) \sin. (S-c) \}}{\sin. b . \sin. c}$$

and, $\log. \sin. A = \frac{1}{2} \{ 20 + 2 \log. 2 + \log. \sin. S + \log. \sin. (S-a) + \log. \sin. (S-b) + \log. \sin. (S-c) \} - \log. \sin. b - \log. \sin. c$.

In order to find $\log. \sin. B$, subtract, in the above form, instead of $\log. \sin. b$ and $\log. \sin. c$, $\log. \sin. a$ and $\log. \sin. c$; and, to find $\log. \sin. C$, subtract, $\log. \sin. a$, and $\log. \sin. b$.

Second Method.

By Proposition 16, page 140,

$$1+\cos. A = 2\frac{\sin. S.\sin.(S-a)}{\sin. b.\sin. c};\ \therefore\ \cos.^2\frac{A}{2} = \frac{\sin. S.\sin.(S-a)}{\sin. b.\sin. c}$$

and $\log.\cos.\frac{A}{2} =$

$$\tfrac{1}{2}\{20+\log.\sin. S+\log.\sin.(S-a)-\log.\sin. b-\log.\sin. c\}.$$

Third Method.

By the same Proposition, in the same page,

$1-\cos. A =$

$$2\frac{\sin.(S-b)\sin.(S-c)}{\sin. b.\sin. c};\ \therefore\ \sin.^2\frac{A}{2} = \frac{\sin.(S-b)\sin.(S-c)}{\sin. b.\sin. c}$$

and $\log.\sin.\frac{A}{2} =$

$$\tfrac{1}{2}\{20+\log.\sin.(S-b)+\log.\sin.(S-c)-\log.\sin. b-\log.\sin. c\}.$$

Fourth Method.

Divide the 3d equation by the 2d, and

$$\frac{\sin.\frac{A}{2}}{\cos.\frac{A}{2}},\ \text{or}\ \tan.\frac{A}{2} = \sqrt{\left(\frac{\sin.(S-b)\sin.(S-c)}{\sin. S.\sin.(S-a)}\right)}$$

and, $\log.\tan.\frac{A}{2} =$

$$\tfrac{1}{2}\{20+\log.\sin.(S-b)+\log.\sin.(S-c)-\log.\sin. S-\log.\sin.(S-a)\}$$

since $\sin. b = \frac{r^2}{\text{cosec.} b}$, $\log.\sin. b = 20-\log.\text{cosec.} b$;

$\therefore$ instead of subtracting log. sin. b, &c. in the above forms, we may (which with certain Tables is a convenient operation) add log. cosec. b — 20, &c.

Example.

$$\begin{array}{rrrr} a = & 50^\circ & 54' & 32'' \\ b = & 37 & 47 & 18 \\ c = & 74 & 51 & 50 \\ \hline & 163 & 33 & 40 \end{array}$$

Numerator computed.

$20 + 2 \log. 2 = 20.6020599$

$$\begin{array}{lrrrlr} S = \dfrac{a+b+c}{2} = & 81^\circ & 46' & 50'' \ldots\ldots & \sin. = & 9.9955158 \\ S - a \ldots\ldots = & 30 & 52 & 18 \ldots\ldots & \sin. = & 9.7102163 \\ S - b \ldots\ldots = & 43 & 59 & 32 \ldots\ldots & \sin. = & 9.8417102 \\ S - c \ldots\ldots = & 6 & 55 & 0 \ldots\ldots & \sin. = & 9.0807189 \\ \cline{6-6} & & & & 2 \,| & 59.2302211 \\ & & & & & 29.6151105 \; [a] \end{array}$$

***B* determined.**

$$\begin{array}{rl} \log. \sin. 50^\circ\ 54'\ 32'' = & 9.8899424 \\ \log. \sin. 74\ \ 51\ \ 50 = & 9.9846660 \\ \hline [c] & 19.8746084 \\ [a] - [c] = & 9.7405021 \\ \therefore B = & 33^\circ\ 22'\ 45'' \end{array}$$

***A* determined.**

$$\begin{array}{rl} \log. \sin. 37^\circ\ 47'\ 18'' = & 9.7872806 \\ \log. \sin. 74\ \ 51\ \ 50 = & 9.9846660 \\ \hline & 19.7719466 \; [b] \\ [a] - [b] = & 9.8431639 \\ \therefore A = & 44^\circ\ 10'\ 40'' \end{array}$$

C determined.

$$\begin{array}{lr} \log. \sin. 50^\circ\ 54'\ 32'' \ldots\ldots\ldots\ldots = & 9.8899424 \\ \log. \sin. 37\ \ 47\ \ 18 \ldots\ldots\ldots\ldots = & 9.7872806 \\ \hline [m] = & 19.6772230 \\ [a] - [m] = & 9.9378875 \end{array}$$

$$\begin{aligned} \therefore C &= 180^\circ - \{60^\circ\ 4'\ 54''\} \\ &= 119\ \ 55\ \ 6, \end{aligned}$$

C is greater than 90°, since

$$\cos. C = \frac{\cos. c - \cos. a \,.\, \cos. b}{\sin. a \,.\, \sin. b}$$

which is negative, since $\cos. c < \cos. a \cos. b$.

Solution of the former Example by the Second Method.

$$\begin{array}{lrr}
 & & 20 = 20 \\
S \quad = 81^\circ\, 46'\, 50'' \ldots\ldots\ldots\ldots\ldots\ldots & \text{sin.} = & 9.9955158 \\
S - a = 30 \;\; 52 \;\; 18 \ldots\ldots\ldots\ldots\ldots\ldots & \text{sin.} = & \underline{9.7102163} \\
 & & 39.7057321 \; [d]
\end{array}$$

$$\begin{array}{lrl}
b = 37^\circ\, 47'\, 18'' \ldots\ldots\ldots\ldots & \text{sin.} = & 9.7872806 \\
c = 74 \;\; 51 \;\; 50 \ldots\ldots\ldots\ldots & \text{sin.} = & \underline{9.9846660} \\
 & & 19.7719466 \ldots\ldots\ldots [b]
\end{array}$$

$$[d] - [b] = 19.9337855$$

$$\tfrac{1}{2}([d] - [b]) = 9.9668927 = \text{log. cos. } \frac{A}{2};$$

$$\therefore \frac{A}{2} = 22^\circ\; 5'\; 20'',$$

$$\text{and } A = 44 \;\; 10 \;\; 40$$

By the Third Method.

$$\begin{array}{lrr}
 & & 20 = 20 \\
S - b = 43^\circ\, 59'\, 32'' \ldots\ldots\ldots\ldots & \text{sin.} = & 9.8417102 \\
S - c = \;\; 6 \;\; 55 \;\;\; 0 \ldots\ldots\ldots\ldots & \text{sin.} = & \underline{9.0807189} \\
 & & 38.9224291 \ldots\ldots\ldots [f] \\
 & [b] = & \underline{19.7719466} \\
 & [f] - [b] = & 19.1504825
\end{array}$$

$$\tfrac{1}{2}([f] - [b]) = 9.5752412 = \text{log. sin. } \frac{A}{2}$$

$$\therefore \frac{A}{2} = 22^\circ\; 5'\; 20'', \text{ and } A = 44^\circ\; 10'\; 40'', \text{ as before.}$$

By the Fourth Method.

$$\begin{array}{lr}
 & 20 = 20 \\
\text{log. sin. } (S - b) + \text{log. sin. } (S - c), \text{ by last, or } [f] = & 38.9224291 \\
\text{log. sin. } S + \text{log. sin. } (S - a), \text{ by 2d solution, or } [d] = & \underline{39.7057321} \\
 & 19.2166970
\end{array}$$

$$\therefore \text{log. tan. } \frac{A}{2} = 9.6083485$$

$$\therefore \frac{A}{2} = 22^\circ\, 5'\, 20'', \text{ and } A = 44^\circ\; 10'\; 40'', \text{ as before.}$$

The solution of this Example becomes the solution of an Astronomical Problem, when from the co-latitude of the place b, or $37^{\circ}\ 47'\ 18''$, the co-declination of the Sun c, or $74^{\circ}\ 51'\ 50''$, and the zenith distance a, or $50^{\circ}\ 54'\ 32''$: the time from noon or the angle A is required; the angle C is the azimuth.

Any one of the four preceding methods may be used, but not, in point of brevity, with equal advantage: if one angle only be required, the shortest solution is plainly by means of one of the three last formulæ: but, if all the angles of the spherical triangle be required, the first method is as short, and quite as convenient, as the three last methods *.

Any one of these four methods may be used, but not, with regard to numerical accuracy, in all cases, with equal advantage. If the angle sought, A for instance, should happen to be nearly 90°, then the first method is to be superseded by one of the three latter, and this for reasons precisely the same, as those which have been stated in page 92, to which the Reader is referred.

CASE 2d, ($ABC\,a$).

The angles A, B, C, are given, and the side a required.

First Method of Solution.

Let the sides and angles of the supplemental or polar triangle be a', b', c'; A', B', C': then, by the last Case,

$$\sin. A' = 2\frac{\sqrt{\{\sin. S.\sin.(S-a').\sin.(S-b')\sin.(S-c')\}}}{\sin. b'.\sin. c'};$$

---

* In the *Logarithmical Arithmetike*, published in 1631, the first case is solved by the second method, and in Vlacq's Tables, published at Gòuda in 1633, the year in which the *Trigonometria Britannica* was published, the solution is by the third method: the rules of solution, then, 200 years ago, were as plain and precise as they now are: yet, in the mode of proof we have gained something, which is certainly more plain and direct than Vlacq's.

but $S = \frac{a' + b' + c'}{2} = \frac{1}{2}(180^\circ - A + 180^\circ - B + 180^\circ - C) =$

$270^\circ - \frac{A + B + C}{2} = (270^\circ - S')$, if $\frac{A + B + C}{2} = S'$;

hence, sin. $S = -$ cos. S'.

Again, $S - a' = 270^\circ - S' - (180^\circ - A) = 90^\circ - (S' - A)$

$\therefore$ sin. $(S - a') =$ cos. $(S' - A)$;

also, $S - b' = 90^\circ - (S' - B)$; $\therefore$ sin. $(S - b') =$ cos. $(S' - B)$; and similarly, sin. $(S - c') =$ cos. $(S' - C)$.

Again, sin. $b' =$ sin. $(180^\circ - B) =$ sin. B; sin. $c' =$ sin. C, and sin. $A' =$ sin. a.

Hence,

$$\text{sin. } a = 2\,\frac{\sqrt{\{-\text{cos. } S'.\text{cos. }(S'-A).\text{cos. }(S'-B).\text{cos. }(S'-C)\}}}{\text{sin. } B.\text{sin. } C}$$

and in logarithms,

log. sin. $a = \frac{1}{2}\{2$ log. $2 + 20 +$ log. cos. $S' +$ log. cos. $(S' - A)$ $+$ log. cos. $(S' - B) +$ log. cos. $(S' - C)\} -$ (log. sin. $B +$ log. sin. C) and sin. b, sin. c, are represented by fractions that have the same numerators as sin. a, and denominators, which are equal to sin. A . sin. C, and sin. A . sin. B, respectively.

The cos. S' under the vinculum is affected with the negative sign. Now, by Prop. 9, page 131, $A + B + C > 180^\circ$, and $< 540^\circ$; $\therefore \frac{A + B + C}{2} > 90^\circ$, and $< 270^\circ$; $\therefore$ cos. S' is negative, or $-$cos. S' is positive. Again, $S' - A = \frac{B + C - A}{2}$: but, by Prop. 2, page 127, $b' + c' > a'$; $\therefore 180^\circ - B + 180^\circ - C > 180^\circ - A$; $\therefore B + C - A < 180^\circ$, and $\frac{B + C - A}{2} < 90^\circ$; $\therefore$ cos. $(S' - A)$ is positive; so are, by similar proofs, cos. $(S' - B)$ and cos. $(S' - C)$.

Hence, in the foregoing expression for sin. a, the quantity under the radical sign is really a positive quantity.

As the expression for sin. A has, by the aid of the supplemental triangle, been employed to represent a side in terms of the angles, so, in like manner, may the expressions for cos. $\frac{A}{2}$, sin. $\frac{A}{2}$, tan. $\frac{A}{2}$; thus by the expression, page 159,

$$\cos.\ \frac{A'}{2} = \sqrt{\left(\frac{\sin.\ S\ .\ \sin.(S-a')}{\sin.\ b'.\ \sin.\ c'}\right)},$$

or, since cos. $\frac{A'}{2} = \cos.\ \frac{(180^0 - a)}{2} = \sin.\ \frac{a}{2}$, we shall have a

Second Method of Solution,

And, a may be found from the expression,

$$\sin.\ \frac{a}{2} = \sqrt{\left(\frac{-\cos.\ S'.\cos.\ (S' - A)}{\sin.\ B\ .\ \sin.\ C}\right)}:$$

and similarly, a

Third Method,

And, a may be found from the expression,

$$\cos.\ \frac{a}{2} = \sqrt{\left(\frac{\cos.\ (S' - B)\ .\ \cos.\ (S-C)}{\sin.\ B\ .\ \sin.\ C}\right)}$$

and similarly, a

Fourth Method,

And, a may be found from the expression,

$$\tan.\ \frac{a}{2} = \sqrt{\left(\frac{-\cos.\ S'\ .\ \cos.\ (S'-A)}{\cos.\ (S' - B)\ .\ \cos.\ (S' - C)}\right)}.$$

Example by the First Method.

a determined.

$A = 44^\circ\ 10'\ 40''$ .... sin. $= 9.8431624$ ...... [1]
$B = 33\ \ 22\ \ 45$ .... sin. $= 9.7405025$ ...... [2]
$C = 119\ \ 55\ \ 6$ .... sin. $= 9.9378874$ ...... [3]

$197\ \ 28\ \ 31$
$\therefore S' = 98\ \ 44\ \ 15.5$ .... cos. $= 9.1815867$
$S - A = 54\ \ 33\ \ 35.5$ .... cos. $= 9.7633172$
$S' - B = 65\ \ 21\ \ 30.5$ .... cos. $= 9.6200732$
$S' - C = -(21\ \ 19\ \ 50.5)$ .... cos. $= 9.9696235$
$20 + 2\ \log.\ 2 =$ 20.6020599

$2)59.1366605$ ...... [*s*]
29.5683302
$[2] + [3] = 19.6783899$

log. sin. $a = 9.8899403$
and $a = 50^\circ\ 54'\ 30''.8$.

b determined.

[*s*] .... $= 29.5683302$
$[1] + [3]$ .... $= 19.7810498$

sin. b .... $= 9.7872804$
and, b .... $= 37^\circ\ 47'\ 18''$.

c determined.

[*s*] .... $= 29.5683302$
$[1] + [2]$ .... $= 19.5836649$

sin. c .... $= 9.9846653$
and c .... $= 74^\circ\ 51'\ 49''$.

By the Second Method.

log. r^2 .... $= 20$
$S' = 98^\circ\ 44'\ 15''.5$ .... cos. $= 9.1815867$
$S' - A = 54\ \ 33\ \ 35.5$ .... cos. $= 9.7633172$

38.9449039
$[2] + [3]$, or log. sin. B + log. sin. C .... $= 19.6783899$

$2)19.2665140$

$\therefore$ log. sin. $\frac{a}{2} = 9.6332570$

$\therefore \frac{a}{2} = 25^\circ\ 27'\ 15''\frac{2}{5}$

and $a = 50\ \ 54\ \ 30\frac{4}{5}$.

By the Third Method.

$$\begin{array}{lr}
\log. r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = & 20 \\
S'-B = 65^\circ\ 21'\ 30''.5 \ldots\ldots\ldots\ldots \cos. = & 9.6200732 \\
S'-C = -21\ \ 10\ \ 50.5 \ldots\ldots\ldots\ldots \cos. = & \underline{9.9696235} \\
 & 39.5896967 \\
{[2]} + {[3]} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \underline{19.6783899} \\
 & 2)19.9113068
\end{array}$$

$$\therefore \log. \cos. \frac{a}{2} = 9.9556534$$

$$\therefore \frac{a}{2} = 25^\circ\ 27'\ 15''\tfrac{2}{5},$$

$$\text{and } a = 50\ \ 54\ \ 30\tfrac{4}{5}.$$

By the Fourth Method.

$$\begin{array}{lr}
\log. r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = & 20 \\
\log. \cos. S' + \log. \cos. (S'-A) \ldots\ldots\ldots = & 18.9449039 \\
\log. \cos. (S'-B) + \log. \cos. (S'-C) \ldots\ldots = & \underline{19.5896967} \\
 & 2)19.3552072
\end{array}$$

$$\log. \tan. \frac{a}{2} = 9.6776036$$

$$\therefore \frac{a}{2} = 25^\circ\ 27'\ 15''\tfrac{2}{5},$$

$$\text{and } a = 50\ \ 54\ \ 30\tfrac{4}{5}.$$

CASE 3d. ($a\,C\,b\,A$).

a, b, two sides and C the included angle are given, and the angle A is required.

By COR. 2. Prop. 16, page 141,

$$\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}; \therefore \frac{\sin. A \pm \sin. B}{\sin. B} = \frac{\sin. a \pm \sin. b}{\sin. b}.$$

By COR. to Prop. 15. $\cos. A + \cos. B = 2. \dfrac{\sin. (a+b)}{\sin. c} \sin.^2 \dfrac{C}{2}$;

$$\therefore \frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \frac{\sin. a + \sin. b}{2 . \sin. (a+b) \sin.^2 \frac{C}{2}} . \frac{\sin. c . \sin. B}{\sin. b};$$

but $\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \tan. \left(\frac{A+B}{2}\right)$ page 37, and

$\frac{\sin. c . \sin. B}{\sin. b} = \sin. C = 2 \sin. \frac{C}{2} \times \cos. \frac{C}{2}$, p. 42 and

$$\therefore \tan. \left(\frac{A+B}{2}\right) = \frac{\sin. a + \sin b}{2 \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)} . \cot. \frac{C}{2}$$

$$= \frac{2 . \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a-b}{2}\right)}{2 . \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)} \cot. \frac{C}{2} \text{ (by form [5], page 32)}$$

$$\text{and } \therefore \tan. \left(\frac{A+B}{2}\right) = \frac{\cos. \left(\frac{a-b}{2}\right)}{\cos. \left(\frac{a+b}{2}\right)} . \cot. \frac{C}{2} \ldots\ldots\ldots\ldots [1]$$

Similarly,

$$\frac{\sin. A - \sin. B}{\cos. A + \cos. B} = \frac{\sin. a - \sin. b}{2 \sin. (a+b) . \sin.^2 \frac{C}{2}} . \sin. C,$$

$$= \frac{2 . \sin. \left(\frac{a-b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)}{2 . 2 . \sin. \left(\frac{a+b}{2}\right) . \cos. \left(\frac{a+b}{2}\right)} 2 \cot. \frac{C}{2};$$

$$\text{or, } \tan. \left(\frac{A-B}{2}\right) = \frac{\sin. \left(\frac{a-b}{2}\right)}{\sin. \left(\frac{a+b}{2}\right)} . \cot. \frac{C}{2} \ldots\ldots\ldots\ldots [2]$$

Hence, since by the formula [1], $\tan. \left(\frac{A+B}{2}\right)$ is determined, $\frac{A+B}{2}$ becomes known, let it $= S$: and, by the formula [2],

$\tan.\left(\frac{A-B}{2}\right)$ is determined, and $\frac{A-B}{2}$ becomes known, let it equal D, then, since

$$\frac{A+B}{2} = S$$

$$\frac{A-B}{2} = D;$$

by addition, $A = S + D$,

by subtraction, $B = S - D$.

Example.

$A +$ determined.

$C = 36° \; 45' \; 28''$ $\qquad \frac{C}{2} = 18° \; 22' \; 44''$....cot.$= 10.4785395$

$a = 84 \;\; 14 \;\; 29$ $\qquad \frac{a-b}{2} = 20 \;\; 0 \;\; 22$.....cos.$= \;\; 9.9729690$

20.4515085

$b = 44 \;\; 13 \;\; 45$ $\qquad \frac{a+b}{2} = 64 \;\; 14 \;\; 7$.....cos.$= \;\; 9.6381663$

$\therefore$ log. tan. $\frac{A+B}{2} = 10.8133422$

$\therefore \frac{A+B}{2} = 81° \; 15' \; 44''.41.$

$A - B$ determined.

$\frac{C}{2} = 18° \; 22' \; 44''$....cot.$= 10.4785395$

$\frac{a-b}{2} = 20 \;\; 0 \;\; 22$...sin. $= \;\; 9.5341789$

20.0127184

$\frac{a+b}{2} = 64 \;\; 14 \;\; 7$....sin. $= \;\; 9.9545255$

10.0581929

$\frac{A-B}{2} = 48° \; 49' \; 38''.$

A and *B* determined.

$$\begin{aligned}
\frac{A+B}{2} &= 81^\circ\ 15'\ 44''.41 \\
\frac{A-B}{2} &= 48\ \ 49\ \ 38 \\
\therefore A &= 130^\circ\ \ 5'\ 22''.41 \\
B &= 32\ \ 26\ \ \ 6.41
\end{aligned}$$

c determined.

$$\begin{aligned}
\text{sin. } 36^\circ\ 45'\ 28'' &= 9.7770158 \\
\text{sin. } 44\ \ 13\ \ 45 &= 9.8435629 \\
&\ \ \ 19.6205787 \\
\text{sin. } 32\ \ 26\ \ \ 6 &= 9.7294422 \\
\therefore \text{log. sin. } c &= 9.8911365 \\
\therefore c &= 51^\circ\ 6'\ 12''.
\end{aligned}$$

The expressions for tan. $\left(\frac{A+B}{2}\right)$ and tan. $\left(\frac{A-B}{2}\right)$, expanded into a proportion, are called, from their Inventor, Naper's *Analogies*.

The angles *A* and *B* being determined by the above forms, the side *c* may be determined, as it has been in the foregoing Example, from the expression $\frac{\text{sin. } c}{\text{sin. } C} = \frac{\text{sin. } b}{\text{sin. } B}$: but it may be desirable, as in the corresponding case of rectilinear triangles (see page 78), to determine *c* immediately without the intervening process of finding the angles *A* and *B*: and, in fact, many Problems in Astronomy * require, from the data of two sides and the included angle, solely the determination of the opposite side *c*.

Determination of the side *c*.

Second Method.

Since $\cos. C = \frac{\cos. c - \cos. a \cdot \cos. b}{\sin. a \cdot \sin. b}$, page 139,

$$\therefore \cos. c = \cos. a \cdot \cos. b + \sin. a \cdot \sin. b \cdot \cos. C;$$

but $\cos. C = 1 - \text{ver. sin. } C$, (ver. sin. stands for versed sine);

---

* For instance, in finding the Moon's distance from a Star; in deducing the altitude of a Star from the latitude, declination and hour-angle (two Problems useful in determining the Longitude); in deducing, in the case of an occultation, the Moon's distance from a Star; in determining the altitude of the nonagesimal (see *Astronomy*, p. 364.): in determining the latitude from 2 altitudes and the time between, (see *Astronomy*, p. 422.), &c.

$$\therefore \cos. c = \cos. a . \cos. b + \sin. a . \sin. b - \sin. a . \sin. b . \text{ver sin. } C$$
$$= \cos. (a - b) - \sin. a . \sin. b . \text{ver. sin. } C;$$

$$\therefore 1 - \cos. c, \text{ or, } 2 \sin.^2 \frac{c}{2} = \text{ver. sin.} (a - b) + \sin. a . \sin. b . \text{ver. sin. } C$$

$$= \text{ver. sin.} (a - b) \left(1 + \frac{\sin. a . \sin. b . \text{ver. sin. } C}{\text{ver. sin.} (a - b)}\right)$$

Assume* $\tan.^2 \theta = \dfrac{\sin. a . \sin. b . \text{ver. sin. } C}{\text{ver. sin. } (a - b)}$;

which in logarithms is $2 \log. \tan. \theta =$

$$\log. \sin. a + \log. \sin. b + \log. \text{ver. sin. } C - \log. \text{ver. sin.} (a - b) \quad [p]$$

then $2 \sin.^2 \dfrac{c}{2} = \text{ver. sin.} (a - b) . \sec.^2 \theta$, and

$$\log. 2 + 2 \log. \sin. \frac{c}{2} = \log. \text{ver. sin.} (a - b) + 2 \log. \sec. \theta - 10 \quad [q]$$

Former Example.

c computed independently of A and B by the 2d Method.

Determination of the subsidiary angle θ by the form [p]

$a = 84^\circ\ 14'\ 29''$ .... sin. = 9.9978028
$b = 44\ \ 13\ \ 45$ .... sin. = 9.8435629
$C = 36\ \ 45\ \ 28$ .... ver. sin. = 9.2984762
29.1398419
$a - b = 40\ \ 0\ \ 44$ .... ver. sin. = 9.3693878
$\therefore$ 2 log. tan. θ .... = 19.7704541
and log. tan. θ .... = 9.8852270

---

* This is the instance to which we alluded in speaking, page 103, of the use of Trigonometrical formulæ in computing log. $(a+b)$.

Determination of c by the form [q], page 170.

$$
\begin{array}{ll}
2\,\text{log. sec. } \theta \ldots\ldots\ldots\ldots\ldots\ldots & = 20.2012488 \\
\text{log. ver. sin. } 40^\circ\ 0'\ 44'' \ldots\ldots\ldots\ldots & = \underline{\ 9.3693878} \\
 & \quad 29.5706366 \\
10 + \text{log. } 2 \ldots\ldots\ldots\ldots\ldots\ldots & = \underline{10.3010300} \\
\therefore\ 2\,\text{log. sin. } \frac{c}{2} \ldots\ldots\ldots\ldots\ldots\ldots & = 19.2696066 \\
\text{and log. sin. } \frac{c}{2} \ldots\ldots\ldots\ldots\ldots\ldots & = \ 9.6348033
\end{array}
$$

$$\therefore\ \frac{c}{2} = 25^\circ\ 33'\ 5''\tfrac{8}{11},$$

$$\text{and } c = 51\quad 6\ 11.5, \text{ nearly.}$$

This is, perhaps, the most commodious form for computing c; for, when we use it, we need not consider whether the fraction $\frac{\text{sin. } a \,.\, \text{sin. } b \,.\, \text{v. sin. } C}{\text{ver. sin. } (a-b)}$ is > or < 1, since tan. θ admits of all degrees of magnitude. It is easy, however, to give another formula of computation, thus:

Third Method of computing c.

$$\text{cos. } c = \text{cos. } a \,.\, \text{cos. } b + \text{sin. } a \text{ sin. } b \,.\, \text{cos. } C$$

$$= \text{cos. } a \,.\, \text{cos. } b + \text{sin. } a \,.\, \text{sin. } b \,.\, \left(2 \text{ cos.}^2 \frac{C}{2} - 1\right)$$

$$= \text{cos } (a+b) + \text{sin. } a \,.\, \text{sin. } b \,.\, 2 \text{ cos.}^2 \frac{C}{2};$$

$$\therefore\ 1 - 2 \,.\, \text{sin.}^2 \frac{c}{2} = \text{cos. } (a+b) + \text{sin. } a \,.\, \text{sin. } b \,.\, 2 \text{ cos.}^2 \frac{C}{2};$$

$$\text{and } \therefore\ 2 \text{ sin.}^2 \frac{c}{2} = 2 \,.\, \text{sin.}^2 \frac{a+b}{2} - 2 \text{ sin. } a \,.\, \text{sin. } b \,.\, \text{cos.}^2 \frac{C}{2}.$$

Let $\text{sin. } a \,.\, \text{sin. } b \,.\, \text{cos.}^2 \frac{C}{2} = \text{sin.}^2 M$;

$$\text{and } \therefore\ \text{log. sin. } M = \tfrac{1}{2}\left(2 \text{ log. cos. } \frac{C}{2} + \text{log. sin. } a + \text{log. sin. } b - 20\right)$$

$$\text{then sin.}^2 \frac{c}{2} = \text{sin.}^2 \frac{a+b}{2} - \text{sin.}^2 M$$

$$= \text{sin.}\left(\frac{a}{2} + \frac{b}{2} + M\right).\text{sin.}\left(\frac{a}{2} + \frac{b}{2} - M\right)$$

by the form [c], p. 31 ; and in logarithms,

$$\text{log. sin.}\frac{c}{2} = \tfrac{1}{2}\left\{\text{log. sin.}\left(\frac{a}{2}+\frac{b}{2}+M\right) + \text{log. sin.}\left(\frac{a}{2}+\frac{b}{2}-M\right)\right\}$$

This is the kind of form which Laplace has employed in his *Mecanique Celeste*, Livre 2, page 227.

Former Example, by the third Method.

$$\frac{C}{2} = 18^\circ\, 22'\, 44''.2 . \text{cos.} = 19.9545254$$

$$a = 84\ \ 14\ \ 29 \ldots . \text{sin.} = \ \ 9.9978028$$

$$b = 44\ \ 13\ \ 45 \ldots . \text{sin.} = \ \ \underline{9.8435629}$$

$$\frac{a+b}{2} = 64\ \ 14\ \ \ 7 \ldots\ldots\ldots\ldots 39.7958911$$

$$M = \underline{52\ \ 14\ \ 23} \ldots\ldots\ldots \therefore \underline{9.8979455} = \text{log.sin.}\, M = \text{log. sin.}\, 52^\circ\, 14'\, 23''$$

$$\therefore\ \frac{a+b}{2} + M = 116^\circ\, 28'\, 30'' \ldots \text{sin.} = 9.9518856$$

$$\frac{a+b}{2} - M = \ \ 11\ \ \ 59\ \ \ 44 \ldots \text{sin.} = \underline{9.3177204}$$

$$19.2696060$$

$$9{,}6348038 = \text{log. sin.}\,\frac{c}{2};$$

$$\therefore\ \frac{c}{2} = 25^\circ\, 33'\, 5''\tfrac{29}{44}$$

$$c = 51\ \ \ 6\ 11.33, \text{ nearly,}$$

Fourth Method.

$$\text{cos.}\, c = \text{cos.}\, a\, .\, \text{cos.}\, b + \text{sin.}\, a\, .\, \text{sin.}\, b\, .\, \text{cos.}\, C$$

$$1 - \text{ver. sin.}\, c = \text{cos.}\, a . \text{cos.}\, b + \text{sin.}\, a\, .\, \text{sin.}\, b - \text{sin.}\, a\, .\, \text{sin.}\, b\, .\, \text{ver. sin.}\, C\, ;$$

$$\begin{aligned}\therefore\ \text{ver. sin.}\, c &= 1 - \text{cos.}\,(a - b) + \text{sin.}\, a . \text{sin.}\, b\, .\, \text{ver. sin.}\, C\\ &= \text{ver. sin.}\,(a - b) + \text{sin.}\, a\, .\, \text{sin.}\, b\, .\, \text{ver. sin.}\, C,\end{aligned}$$

which formula, translated into words, becomes the precept given in Sherwin's Table, page 44, (edit. 1771) for finding the side opposite to the included angle. The Author gives in his Work a Table of natural versed sines, which are plainly necessary in his mode of computation.

If we use one of the preceding formulæ for computing c, we may, if we please, determine A and B without the aid of Naper's *Analogies*, and by these expressions,

$$\sin. A = \frac{\sin. C . \sin. a}{\sin. c}; \quad \sin. B = \frac{\sin. C . \sin. b}{\sin. c}.$$

CASE IV. $(AcBa)$.*

A, B, two angles, and c the adjacent side are given, and the side a is required.

The solution will be deduced from the former, by the aid of the supplemental triangle; A', B', C', a', b', c' being its angles and sides.

$$\tan.\left(\frac{A'+B'}{2}\right) = \tan.\left(\frac{(180^0-a)+(180^0-b)}{2}\right) = \tan.\left(180^0 - \frac{a+b}{2}\right)$$

$$= -\tan.\left(\frac{a+b}{2}\right)$$

$$\tan.\left(\frac{A'-B'}{2}\right) = \tan.\left(\frac{b-a}{2}\right) = -\tan.\left(\frac{a-b}{2}\right)$$

$$\sin.\left(\frac{a'+b'}{2}\right) = \sin.\left(180^0 - \frac{A+B}{2}\right) = \sin.\left(\frac{A+B}{2}\right)$$

[see Cor. 6, Prob. 3.]

$$\sin.\left(\frac{a'-b'}{2}\right) = \sin.\left(\frac{B-A}{2}\right) = -\sin.\left(\frac{A-B}{2}\right);$$

---

* This combination is in the third row (see p. 155,) and therefore not essentially different from the first of that row, namely, $(aCbA)$ which has been already considered. In fact, the two equations involving the four quantities are precisely similar: but, from what was said in p. 157, the same formula of solution cannot suit each case, since in the former, an angle is the quantity required, which in the latter, becomes one of the quantities given.

$$\therefore \text{ generally, } \sin.\left(\frac{a'\pm b'}{2}\right) = \pm \sin.\left(\frac{A\pm B}{2}\right).$$

Again,

$$\cos.\frac{a'+b'}{2} = \cos.\left(180^\circ - \frac{A+B}{2}\right) = -\cos.\left(\frac{A+B}{2}\right)$$

[see Cor. 6, Prob. 3.]

$$\cos.\left(\frac{a'-b'}{2}\right) = \cos.\left(\frac{B-A}{2}\right); \text{ or } = \cos.\left(\frac{A-B}{2}\right);$$

$$\text{lastly, } \cot.\frac{C'}{2} = \cot.\left(90^\circ - \frac{c}{2}\right) = \tan.\frac{c}{2}.$$

Having now transformed all the terms in [1], [2], (see p. 167), into different expressions, if we substitute the transformed terms in [1], [2], there results

$$\tan.\left(\frac{a+b}{2}\right) = \frac{\left(\cos.\frac{A-B}{2}\right)}{\left(\cos.\frac{A+B}{2}\right)}.\tan.\frac{c}{2}\ldots\ldots\ldots[3],$$

$$\text{whence } \frac{a+b}{2},$$

and

$$\tan.\left(\frac{a-b}{2}\right) = \frac{\sin.\left(\frac{A-B}{2}\right)}{\sin.\left(\frac{A+B}{2}\right)}\tan.\frac{c}{2}\ldots\ldots\ldots\ldots[4],$$

whence $\frac{a-b}{2}$ may be derived.

These equations [3], [4], expanded into a proportion, are called, [as the former similar ones [1], [2] have been (see p. 169.)] Naper's *Analogies*.

Example.

$$c = 51^\circ\ 6'\ 12''$$
$$A = 130\ \ 5\ \ 22$$
$$B = 32\ \ 26\ \ 6$$

$$\frac{A-B}{2} = 48\ \ 49\ \ 38 \ldots\ldots \text{cos.} = 9.8184449 \ldots\ldots \text{sin.} = 9.8766379$$

$$\frac{c}{2} = 25\ \ 33\ \ 6 \ldots\ldots \text{tan.} = \underline{9.6795032} \ldots\ldots \text{tan.} = \underline{9.6795032}$$

$$19.4979481 \qquad\qquad 19.5561411$$

$$\frac{A+B}{2} = 81\ \ 15\ \ 44 \ldots\ldots \text{cos.} = \underline{9.1815936} \ldots\ldots \text{sin.} = \underline{9.9949301}$$

$$\text{tan.}\,\frac{a+b}{2} = 10.3163545 \qquad \text{tan.}\,\frac{a-b}{2} = 9.5612110$$

$$\text{and}\ \frac{a+b}{2} = 64^\circ\ 14'\ 6''\tfrac{1}{8} \qquad \text{and}\ \frac{a-b}{2} = 20^\circ\ 0'\ 22''\tfrac{2}{13}$$

$$\frac{a-b}{2} = \underline{20\ \ \ 0\ 22\tfrac{2}{13}}$$

$$\therefore\ a = 84^\circ\ 14'\ 28''\tfrac{29}{104}$$

$$b = 44\ \ 13\ \ 43\tfrac{101}{104}.$$

C determined.

$$\text{sin. } 130^\circ\ 5'\ 22'' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 9.8836842$$
$$\text{sin. } 51\ \ 6\ \ 12 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \underline{9.8911357}$$
$$19.7748199$$
$$\text{sin. } 84\ 14\ 29 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \underline{9.9978028}$$
$$\therefore\ \text{log. sin. } C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 9.7770171$$

$$\therefore\ C = 36^\circ\ 45'\ 28''.$$

a and b being determined, C may be determined from this expression, $\text{sin.}\,C = \text{sin.}\,A \times \dfrac{\text{sin.}\,c}{\text{sin.}\,a}$, as it was in the preceding Example, or, without the intervening process of finding a and b, by the following method.

Determination of the angle C.

In the supplemental triangle, by the original form [c], p. 139,

$$\cos.\, c' = \sin.\, a' . \sin.\, b' . \cos.\, C' + \cos.\, a' . \cos.\, b' ;$$

$$\text{or } \cos.\,(180^{\circ} - C) = \sin.\,(180^{\circ} - A) . \sin.\,(180^{\circ} - B) . \cos.\,(180^{\circ} - c)$$

$$+ \cos.\,(180^{\circ} - A) \cos.\,(180^{\circ} - B)$$

$$\text{or } - \cos.\, C = - \sin.\, A . \sin.\, B . \cos.\, c + \cos.\, A . \cos.\, B ;$$

$$= - \sin.\, A . \sin.\, B\, (1 - \text{ver. sin. } c) + \cos.\, A . \cos.\, B ;$$

and $\therefore$ $1 - \cos.\, C$, or, ver. sin. $C =$

$$1 + \cos.\, A . \cos.\, B - \sin.\, A . \sin.\, B + \text{ver. sin. } c . \sin.\, A . \sin.\, B$$

$$= 1 + \cos.\,(A + B) + \text{ver. sin. } c . \sin.\, A \sin.\, B ;$$

$$\therefore \text{ ver. sin. } C, \text{ or } 2 \sin.^{2} \frac{C}{2} =$$

$$2 . \cos.^{2} \left(\frac{A + B}{2} \right) + \text{ver. sin. } c . \sin.\, A \sin.\, B =$$

$$. \cos.^{2} \frac{A + B}{2} \left(1 + \frac{\text{ver. sin. } c . \sin.\, A . \sin.\, B}{2 . \cos.^{2} \left(\frac{A + B}{2} \right)} \right) .$$

Assume $\tan.^{2} \theta = \dfrac{\text{ver. sin. } c . \sin.\, A . \sin.\, B}{2 . \cos.^{2} \left(\frac{A + B}{2} \right)}$, or in logarithms,

$\log.\, \tan.\, \theta =$

$$\tfrac{1}{2} \left(\log.\, \text{ver. sin. } c + \log.\, \sin.\, A + \log.\, \sin.\, B - 2 \log.\, \cos.\, \left(\frac{A + B}{2} \right) + 10 - \log.\, 2 \right).$$

then, ver. sin. C, or, $2 \sin^{2} . \dfrac{C}{2} = 2 . \cos.^{2} \left(\dfrac{A + B}{2} \right) \sec.^{2} \theta$;

in logarithms, $\log.\, \sin.\, \dfrac{C}{2} = \log.\, \cos.\, \dfrac{A + B}{2} + \log.\, \sec.\, \theta - 10.$

C, in the former Example, found independently of a, b.

$$\begin{array}{lllll}
 & & & 10 - \text{log. } 2 = & 9.6989690 \\
c = 51^\circ & 6' & 12'' \ldots\ldots \therefore & \text{ver. sin.} = & 9.5706390\,^* \\
A = 130 & 5 & 22 \ldots\ldots\ldots\ldots & \text{sin.} = & 9.8836842 \\
B = 32 & 26 & 6 \ldots\ldots\ldots\ldots & \text{sin.} = & \underline{9.7294422} \\
 & & & & 38.8827344 \\
\frac{A+B}{2} = 81 & 15 & 44 \ldots\ldots\ldots & 2\,\text{cos.} = & \underline{18.3631872} \ldots\ldots \text{cos.} = 9.1815936 \\
 & & & & 20.5195472 \\
 & & & \therefore \text{log. tan. } \theta = & 10.2597736 \ldots \text{sec.}\,\theta = \underline{10.3171290} \\
 & & & & \qquad \therefore \text{log. sin. } \frac{C}{2} = 9.4987226 \\
 & & & & \qquad \therefore \frac{C}{2} = 18^\circ\ 22'\ 43''\tfrac{4}{7},
\end{array}$$

and $C = 36^\circ\ 45'\ 27''$ nearly, as before.

If we express $1 + \text{cos.}\,(A + B)$, the versed sine of the supplement of $A + B$, by suver. sin. $(A + B)$ we may employ this form for computing C

$$\begin{aligned}
\text{ver. sin. } C &= \text{suver. sin. } (A + B) + \text{ver. sin. } c \,.\, \text{sin. } A \,.\, \text{sin. } B \\
&= \text{suver. sin. } (A + B) \,.\, \text{sec.}^2\,\theta, \text{ putting}
\end{aligned}$$

$$\text{tan.}^2\,\theta = \text{sin. } A \,.\, \text{sin. } B \,.\, \frac{\text{ver. sin. } c}{\text{suver. sin. } (A + B)}.$$

CASE V. $(a\,A\,b\,B)$.

Two sides a, b, and an angle A opposite to one, are given, the angle B, and the remaining angle and side are required.

By Cor. 2, page 141, $\text{sin. } B = \dfrac{\text{sin. } A \,.\, \text{sin. } b}{\text{sin. } a}$, and in order to find C, take the first of Naper's *Analogies*, p. 167, then $\cot. \frac{C}{2} = \tan. \frac{1}{2}(A + B) \frac{\cos. \frac{1}{2}(a + b)}{\cos. \frac{1}{2}(a + b)}$ .... [a]

---

* Since $\text{ver. sin. } c = 2\,\text{sin.}^2 \frac{c}{2}$, $\text{log. ver. sin. } c - \text{log. } 2 = 2\,\text{log. sin. } \frac{c}{2} - 10$

$\therefore \text{log. tan. } \theta = \frac{1}{2}\left(2\,\text{log. sin. } \frac{c}{2} + \text{log. sin. } A + \text{log. sin. } B - 2\,\text{log. cos. } \frac{A+B}{2}\right)$

which form is rather more convenient than the one used in the computation.

and $\log. \cot. \frac{C}{2} =$

$\log. \tan. \frac{1}{2}(A + B) + \log. \cos. \frac{1}{2}(a + b) - \log. \cos. \frac{1}{2}(a - b)$.

C being found, c may be had from the expression

$\sin. c = \sin. a . \frac{\sin. C}{\sin. A}$, or directly thus from the third of Naper's *Analogies*, p. 174,

$$\tan. \frac{c}{2} = \tan. \frac{1}{2}(a + b) \frac{\cos. \frac{1}{2}(A+B)}{\cos. \frac{1}{2}(A-B)} \ldots\ldots\ldots\ldots \quad [b]$$

Example. B computed.

$A = 33° 15' 7''$ .... sin. = 9.7390354
$b = 70\ 10\ 30$ .... sin. = 9.9734663
19.7125017
$a = 80\ 5\ 4$ .... sin. = 9.9934638
log. sin. B .... sin. = 9.7190379

and $B = 31° 34' 37''.71$.

C computed.

$\frac{a+b}{2} = 75° 7' 47''$ .... cos. = 9.4093099
$\frac{A+B}{2} = 32\ 24\ 52$ .... tan. = 9.8027553
19.2120652
$\frac{a-b}{2} = 4\ 57\ 17$ .... cos. = 9.9983741

$\cot. \frac{C}{2} = 9.2136911$

$\therefore \frac{C}{2} = 80° 42' 38''\frac{7}{11}$,

and $C = 161\ 25\ 17\frac{3}{11}$.

c computed from *C*.

sin. 161° 25′ 17″ ………………………… = 9.5032532
sin. 80 5 4 ………………………… = 9.9934638

19.4967170
sin. 33 15 7 ………………………… = 9.7390354

log. sin. *c* ………………………… = 9.7576816

$\therefore c = 145^\circ\ 5'\ 2''$.

c computed independently of the value of *C*.

$\frac{a+b}{2} = 75^\circ\ 7'\ 47''$ ……………… tan. = 10.5758962
$\frac{A+B}{2} = 32\ 24\ 52$ ……………… cos. = 9.9264417

20.5023379
$\frac{A-B}{2} = 0\ 50\ 15$ ……………… cos. = 9.9999536

$\therefore \tan. \frac{c}{2} = 10.5023843$

$$\therefore \frac{c}{2} = 72^0\ 32''\ 30''\tfrac{7}{12},$$

$$\text{and } c = 145\ \ 5\ \ 1\tfrac{1}{6}.$$

CASE VI. (*AaBb*).

Two angles *A*, *B* and a side (*a*) opposite to one of them, are given, the other side *b*, and, besides, the remaining side *c* and the angle *C*, are required.

b is determined from this expression, $\sin. b = \sin. a \frac{\sin. B}{\sin. A}$; *C* from the first of Naper's *Analogies* [1], p. 167, and *c* from the third [3], p. 164, as in the preceding case.

b computed.

a = 89° 16′ 53″.5 .... sin. = 9.9999658
B = 48 36 0 .... sin. = 9.8751256
19.8750914
A = 70 39 0 .... sin. = 9.9747475
log. sin. b = 9.9003439

$\therefore b = 52° 39' 4''.5.$

The sin. b = sin. (180° − b), but b cannot = 127° 20′ 56″, for since $A > B$. a must be $> b$.

c computed from the form [b].

$\frac{a+b}{2}$ = 70° 57′ 59″ .... tan. = 10.4622011
$\frac{A+B}{2}$ = 59 37 30 .... cos. = 9.7038563
20.1660574
$\frac{A-B}{2}$ = 11 1 30 .... cos. = 9.9919097
log. tan. $\frac{c}{2}$ = 10.1741477

$\therefore \frac{c}{2} = 56° 11' 29''.33,$

* and c = 112 22 58.6.

---

* This last Example is taken from the *Trigonometry*, of M. Legendre, who has, however, found c and C by a process different from the above. Subjoined are the data and results in French measure (F) and reduced by the Rule, page 21, to English (E).

C computed from the form [a].

$$\begin{array}{ll}
\frac{a+b}{2} = 70^\circ\ 57'\ 59'' \ldots\ldots\ldots\ldots\ldots\ldots \cos. = & 9.5133811\\
\frac{A+B}{2} = 59\ \ 37\ \ 30 \ldots\ldots\ldots\ldots\ldots\ldots \tan. = & 10.2320208\\
\cline{2-2}
 & 19.7454019\\
\frac{a-b}{2} = 18\ \ 18\ \ 54.5 \ldots\ldots\ldots\ldots\ldots\ldots \cos. = & 9.9774230\\
\cline{2-2}
\qquad\qquad\qquad\qquad\qquad \cot.\frac{C}{2} = & 9.7679789
\end{array}$$

$$\therefore\ \frac{C}{2} = 59^\circ\ 37'\ 30'',$$

$$\text{and } C = 119\ \ 15\ \ \ 0.$$

$$\begin{array}{r}
a = 99^\circ\ 20'\ 17''\ (F)\\
9\ \ 92\ \ 017\\
\hline
89\ \ 28\ \ 153\\
6\\
\hline
16.8918\\
6\\
\hline
53.508
\end{array}$$

$$\therefore\ a = 89^\circ\ 16'\ 53''.5\ (E)$$

$$\begin{array}{r}
B = 54^\circ\ 0'\ (F)\\
5\ \ \ 4\\
\hline
48\ \ \ 6\\
6\\
\hline
36
\end{array}$$

$$\therefore\ B = 48^\circ\ 36'\ (E)$$

$$\begin{array}{r}
A = 78^\circ\ 50'\ (F)\\
7\ \ 85\\
\hline
70\ \ 65\\
6\\
\hline
39\ \ \ 0
\end{array}$$

$$\therefore\ A = 70^\circ\ 39'\ (E)$$

$$\begin{array}{r}
c = 124^\circ\ 86'\ 99''.3,\\
12\ \ 48\ \ 69.93\\
\hline
112\ \ 38\ \ 29.37\\
6\\
\hline
22.97622\\
6\\
\hline
58.5732
\end{array}$$

$$\begin{array}{r}
b = 58^\circ\ 50'\ 14''\ (F),\\
5\ \ 85\ \ \ 014\\
\hline
52\ \ 65\ \ 126\\
6\\
\hline
39.0756\\
6\\
\hline
4.536
\end{array}$$

$$\begin{array}{r}
C = 132^\circ\ 50'\\
13\ \ 25\\
\hline
119\ \ 25\\
6\\
\hline
15.0
\end{array}$$

$$\therefore\ C = 119^\circ\ 15'\ (E)$$

$$\therefore\ c = 112^\circ\ 22'\ 58''.6\ (E) \ldots\ldots \therefore\ b = 52^\circ\ 39'\ 4''.5\ (E)$$

and these quantities (c, b, C,) agree with those determined in the text.

The above reductions may, more easily, be performed by the aid of the Table inserted at the end of Chap. I, p. 23.

In Case 5th, the angle C has been determined by means of B previously determined, and by the aid of one of Naper's *Analogies*; and this method, on the grounds of facility and certainty, is, perhaps, the most convenient: still, analytically considered, the determination of C does not require the previous determination of B, for by the third equation, p. 157,

$$\cot. A . \sin. C = \cot. a . \sin. b - \cos. C . \cos. b,$$

in which A, a, b, C are alone involved. But, this form is not adapted to logarithmic computation; in order to adapt it, we must introduce what has been called a *subsidiary* angle: thus, if we take θ such, that $\tan. \theta = \cos. b . \tan. A$ [c]; then,

$$\frac{\sin. C . \cos. b}{\tan. \theta} + \cos. b . \cos. C = \cot. a . \sin. b;$$

$$\text{or, } \cos. b . (\sin. C . \cos. \theta + \cos. C . \sin. \theta) = \cot. a . \sin. b . \sin. \theta;$$

$$\text{or, } \sin. (C + \theta) = \frac{\cot. a . \sin. \theta . \sin. b}{\cos. b} = \tan. b . \sin. \theta . \cot. a \ldots [d]$$

Hence, by deducing the logarithm of $C + \theta$, we shall know C, since θ is determined by this form,

$$\log. \tan. \theta = \log. \cos. b + \log. \tan. A - 10,$$

and by a similar method, that is, by the assumption of a subsidiary angle, may c be determined solely from A, a, b. It is sufficient, however, to have noted these methods, for, the computist is not recommended to avail himself of them; the preceding ones, those by which the Examples have been numerically solved, being fully adequate to the purpose of solution.

In Case 6th, C and c may be also solved by the introduction of a subsidiary angle; and its introduction, in these cases, corresponds to the Geometrical resolution of the oblique-angled into two right-angled triangles: thus, in the last case, conceive a perpendicular (p) the arc of a great circle, to be drawn from the angle C on the base c, and let the angle contained between this perpendicular and the side b be supposed equal to $(90^\circ - \theta)$; then, by Naper's first Rule,

$$1 \times \cos. b = \cot. (90^\circ - \theta) . \cot. A,$$

and $\tan. \theta = \cos. b . \tan. A$, which agrees with the assumption [c], p. 182.

Again, by Naper's first Rule,

$$\sin. \{ 90^\circ - (90^\circ - \theta) \} \text{ or, } \sin. \theta = \tan. p . \cot. b, \text{ and}$$

$$\therefore \tan. p = \sin. \theta . \tan. b.$$

And finally, by Naper's first Rule,

$$\cos. \{ C - (90^\circ - \theta) \} = \tan. p . \cot. a, \text{ or}$$

$$\therefore \sin. (C + \theta) = \sin. \theta . \tan. b . \cot. a,$$

which agrees with the result [d] in the preceding page.

A great variety of instances to the preceding methods might easily be collected from *Plane Astronomy*. It is not, however, necessary to give any; since, amongst other purposes, the present Treatise is meant to be merely preparatory and subservient to the study of the latter science, and to be intelligible to the Student who may happen to be unacquainted with its technical terms and language. Astronomical Examples, stated and numerically resolved, would, indeed, be useful to the Student. One part of their utility would be, to communicate the art of translating Astronomical conditions into bare Mathematical conditions; it is not, however, the special business of a Trigonometrical Treatise to teach such art. Another part of their utility would consist in teaching the method of transforming general symbolical results and formulæ into numerical values; but, of this method sufficient specimens, it is hoped, have been given in the preceding pages.

Still, however, it is desirable to apply and illustrate the preceding formulæ; and, it happens fortunately, we can effect this without introducing either the principles or the terms of a new science. The accounts of those *Trigonometrical Surveys*, by which the figure and dimensions of the Earth have been attempted to be determined, will furnish us with very interesting instances of exemplification.

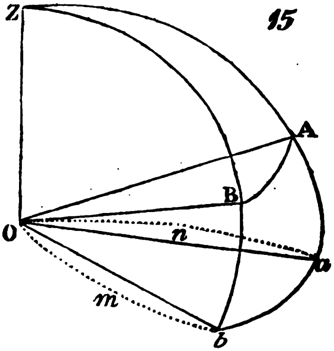
In the next Chapter we will turn our attention to this point. We shall there perceive how results may be obtained by the direct application of the preceding methods of solution; and, besides, for what reasons and by what means, those methods, in certain circumstances, are either modified, or are completely superseded by methods of approximation.

CHAP. XII.

Application of Formulæ to certain Cases that occur in the Trigonometrical Surveys.—Reduction of the Oblique to an Horizontal Angle, by the direct, by Delambre's approximate, Method.—Excess of the Sum of the Three Angles of a Spherical Triangle above Two Right Angles.—Roy's Theorem.—Legendre's Reduction of a small Spherical to a Rectilinear Triangle.—Delambre's and Mudge's Method of computing the Length of the Meridian.

In the Trigonometrical Survey, the two extremities of the line to be measured, are connected by a series of triangles, the angular points of which are the stations, at which, by proper instruments, the angles are observed. The sides of the triangles are determined by computation. But, since the sides cannot be determined solely from the angles, it is necessary to measure, at least, one side of one of the series of triangles. It is usual to measure the length of a side of the first triangle, near one extremity of the line to be measured; which side is called the *Base*; and also another line, the side of one of the triangles, near the other extremity; which latter line, serving as a test of the accuracy of the observations and computations, is called a *Base of Verification*. In the Trigonometrical Survey conducted in 1784, &c. under General Roy, the difference between the base of verification on Romney Marsh, *measured* and *computed* from the original base measured on Hounslow Heath was found to be about 2 feet in 28533 feet. The connecting chain of triangles by which the computation was made, extended over a space of eighty miles.

After the first important operation of the measurement of a base, and which, (the side of the first triangle,) will be a spherical arc, if the measurement be conducted by spherical triangles; or, a straight line, if by plane triangles of the chords of the spherical arcs, the angular distances between certain objects that mark the several stations must be observed. But as the objects, probably, are not situated on the same horizontal plane, nor on the same spherical surface, the angles observed are oblique angles. Since, however, the lines, &c. to be computed, are supposed to lie in the surface of the earth, they are horizontal angles which are required. The oblique angle observed then must be reduced to an horizontal angle: for instance,—if the two objects be *A, B*; *O* the observer, and *Z* his zenith, and *a, b,* two points on the surface of the earth (supposed to be spherical), the observed or oblique



angle, is *AOB,* measured by *AB,* and the horizontal angle, which is required, is equal to *AZB.*

Now, by the third method of solution, Case 1, page 159,

$$\sin.^2 \frac{AZB}{2} = \frac{\sin. \frac{1}{2}(AB+ZB-ZA) . \sin. \frac{1}{2}(AB+ZA-ZB)}{\sin. ZA . \sin. ZB}$$

$$= \frac{\sin. \frac{1}{2}(a+H-h) \sin. \frac{1}{2}(a+h-H)}{\cos. H . \cos. h}, \text{ if } AB=a,\ Aa=H,\ Bb=h$$

and in logarithms,

$$\log. \sin. \frac{AZB}{2} = \tfrac{1}{2} \{ 20 + \log. \sin. \tfrac{1}{2}(a+H-h) + \log. \sin. \tfrac{1}{2} (a+h-H) - \log. \cos. H - \log. \cos. h \}$$

From this formula, a, h, H, being determined by observation, the horizontal angle may be computed. For example: in the Trigonometrical Survey, of 1787, made for the purpose of joining the Observatories of Greenwich and Paris, the angular distance of Watten and Fiennes, at Calais, was observed to be 66° 30′ 38″.9: moreover, H was $= -1''$ (a depression) and $h = 25' \, 47''.2$: hence, if O represent Calais, A and B Watten and Fiennes, $AB = 66^\circ \, 30' \, 38''.9$.

$$\log. r^2 = 20$$

$$\tfrac{1}{2}(a+H-h) = 33^\circ \; 2' \, 25''.35 \ldots\ldots\ldots \sin. = 9.7365802$$

$$\tfrac{1}{2}(a+h-H) = 33 \;\; 28 \;\; 13.55 \ldots\ldots\ldots \sin. = \underline{9.7415519}$$

$$39.4781321$$

$$\log. \cos. -1'' + \log. \cos. 25' \, 47''.2 \ldots\ldots\ldots = \underline{19.9999878}$$

$$\therefore \log. \sin. \frac{AZB}{2} = \tfrac{1}{2}(19.4781443)$$

$$= \quad 9.7390721$$

$$\therefore \frac{AZB}{2} = 33^0 \, 15' \, 18''\tfrac{14}{32}, \text{ and } AZB = 66^\circ \, 30' \, 36''\tfrac{7}{8}.$$

Here, the reduced or horizontal angle differs from the oblique or observed angle by a quantity a little more than 2″; and although the preceding process of reduction is not a long process, yet, for practical purposes, it is not sufficiently short, when several hundred reductions are to be made; and for this cause, M. Delambre * has investigated a formula, and constructed Tables of easy use, for finding a small correction; which, applied to the oblique angle, should give the horizontal angle. A formula similar to Delambre's, will be investigated in the Appendix.

But, whether we use the preceding direct method, or Delambre's approximate method, all observed angles, if it is proposed to conduct the operation with nicety, must be reduced to horizontal angles;

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* See *Connoissance des Temps,* 1793; Maskelyne, *Phil. Trans.* 1797, page 451; Legendre, *Mem. Acad.* 1787, page 313; Delambre, *Mem. Inst.* 1806, p. 112; Suanberg, *Exposition des Operations faites,* &c., p. 38.

and these latter angles are spherical angles. For, if we recur to the preceding method, the angle reduced was *AZB*, which is equal to the inclination of the planes *OZa*, *OZb*. Hence, if from *A* and *B*, as stations, the angles subtended by *BO*, *AO*, respectively, be observed, and reduced, by the preceding method, to horizontal angles, the three stations will be projected on the surface of a sphere, and the sum of the three angles at *O*, *a*, and *b*, ought to exceed 180°. (Prop. 9, p. 131).

If we could determine, independently of observation, the quantity by which the sum of the three angles ought to exceed 180°, we should then be able to judge of the accuracy of observations; which it is desirable to do, since observations made on objects situated nearly in the horizon, are liable to some uncertainty. Now, the Theorem concerning the area of a spherical triangle, (Prop. 13, page 134), enables us to determine the quantity of the excess For, the area is equal to the difference of the sum of the three angles and 180°, and consequently, since the sides of the spherical triangle described on the Earth's surface are nearly rectilinear, they may, with scarcely any error, be considered as rectilinear in the computation of the area.

This beautiful application of Albert Girard's Theorem was first made by General Roy, in the *Philosophical Transactions* for 1790, page 171, where he gave the following rule* for computing, what he terms, the spherical excess. "*From the logarithm of the area of the triangle, taken as a plane one, in feet, subtract the constant logarithm* 9.3267737, *and the remainder is the logarithm of the excess above* 180°, *in seconds nearly.*"

The rule may be thus investigated,

$a + b + c - 180^\circ = \frac{4x}{S} 180^\circ$ (Prop. 13), but $S = 4\pi r^2$; $\therefore a + b + c - 180^\circ = \frac{180^\circ}{\pi r^2} x$; or, if a, b, c, are reduced to seconds, the excess in seconds $= \frac{180.60.60}{\pi r^2} x$; now, on the Earth's surface, 1° (taking a mean measurement) corresponds to $(60859 . 1) \times 6$ feet; $\therefore$ since an arc ($=$ radius) $= \frac{360}{2\pi}$,

---

* M. Delambre finds, very expeditiously, the spherical excess by means of a chart of the triangles, a scale, and a Table of two pages. See *Mem. Inst.* 1806, pages 166, 179, 180. *Disc. Prel.*

$1^\circ : (60859 . 1) \times 6 :: \frac{360}{2\pi} : \frac{360}{2\pi} \times (60859 . 1) \times 6 =$ Earth's radius in feet.

Hence the excess in seconds

$$= x . \frac{2\pi}{360} . \frac{60 . 60}{6^2 \times (60859 . 1)^2}$$

$$= x . \frac{2\pi \times 10}{36 \times (60859 . 1)^2} ;$$

$\therefore$ log. excess

$= \log. x - \{ 2 (\log. 6 + \log. 60859.1) - \log. 2\pi . 10 \}$

$= \log. x - (11.1249536 - 1.7981799)$

$= \log. x - 9.3267737.$

As an instance of this rule, take the Example given by General Roy, *Phil. Trans.* 1790, p. 172.

Names of the Stations.	Observed Angles.	Distances in Feet.
Hanger-Hill Tower......(a)	42° 2′ 32″	(a) from (b) = 38461.12
Hampton Poor-house....(b)	67 55 39	(a) from (c) = 24704.7
King's Arbour.............(c)	70 1 48	
	179 59 59	

Hence, making the distance from (a) to (c) the base of the triangle, the perpendicular on that base will be equal to $38461.12 \times \sin. 42^\circ 2' 32''$, and therefore the area of the triangle $= \frac{\text{base} \times \text{perp}^{\text{r}}.}{2}$

$= 24704.7 \times 19230.56 \times \sin. 42^\circ 2' 32''.$

Computation.

log. sin. 42° 2′ 32″ = 9.8258661
log. 24704.7 = 4.3927761
log. 19230.56 = 4.2839906
18.5026328

∴ taking away 10 for radius, 8.5026328 the logarithm of the area in feet.

8.5026328
9.3267737
−1.1758591

Corr^s. No. = .14992 *, or the spherical excess in seconds is nearly .15.

Again, as a second instance, take that given in the succeeding page;

Names of the Stations.	Observed Angles.	Distances in Feet.
Hundred Acres……(*d*)	53° 58′ 35″.75	
Hanger Hill Tower (*e*)	68 24 44	(*d*) from (*e*) = 71934.2
St. Ann's Hill……..(*f*)	57 36 39.5	(*d*) from (*f*) = 79211.22
	179 59 59.25	

log. sin. 53° 58′ 35″.75 = 9.9078287
log. 35967.1 ………. = 4.5559054
log. 79211.22 ……… = 4.8987866
19.3625207

9.3625207
9.3267737
0.0357470 = log. 1.0858

or spherical excess in seconds = 1″.0858.

Third Instance, from the *Phil. Trans*. 1803. page 428,

Buxter Hill………(*g*)	76° 12′ 22″	
Dean Hill………..(*h*)	48 4 32.25	(*i*) from (*g*) = 140580.4
Dünnose…………..(*i*)	55 43 7	(*i*) from (*h*) = 183496.2

log. sin. 55° 43′ 7″ = 9.9171279
log. 70290.2 …… = 4.8468947
log. 18349.6 …….. = 5.2636266
20.0276492

10.0276492
9.3267737
.7008755 = log. 5″.022;

or spherical excess = 5″.022.

---

* There appears to be an error in the *Philosophical Trans*. p. 172. where the spherical excess is put down .29, which is nearly equal to 2.(14992). The cause of the error seems to be, the neglect of the divisor 2 in estimating the area of the triangle; for if we add the log. 2 to − 1.1758591, the resulting number expresses the logarithm of .29.

In this manner the spherical excess computed*, will enable the observer to examine the accuracy of his observations, and in some degree to correct them. He may then proceed to calculate the sides of the triangles by the rules of Spherical Trigonometry. But these rules, although they must give exact results, will not give results very expeditiously. It is, however, very desirable to obtain results expeditiously, when several hundred triangles are to be solved; and, this desideratum, M. Legendre has put us in possession of, by combining sufficient exactness with conciseness in the following Theorem: "*A spherical triangle being proposed, of which the sides are very small relatively to the radius of the sphere, if from each of its angles one-third of the excess of the sum of its three angles above two right angles be subtracted, the angles so diminished may be taken for the angles of a rectilinear triangle, the sides of which are equal in length to those of the proposed spherical triangle.*" The demonstration of this rule will be given in the Appendix, but its utility may be here shewn. Suppose in a spherical triangle, two sides and the included angle to have been determined, respectively, equal to 53° 58′ 35″.75, 71934.2, 79211.22 feet; compute the spherical excess, which is equal to 1″.08 nearly (see the second Example, p. 190), then $\frac{1}{3}$ (1″.08) = .36, and 53° 58′ 35″.75 − .36 = 53° 58′ 35″.39: with this, as an included angle, and the given values of the two sides, resolve the triangle as a rectilinear triangle, by the method † given in page 83, Plane Trigonometry.

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* In two of the instances that have been given, the sum of three observed angles is less than 180°; ∴ in order to correct that *sum*, we must add the defect plus the spherical excess; that is, in the first instance, we must add 1″ + .15; in the second, .75 + 1″.08. If all the observed angles were made under similar circumstances, and with equal care, we should have angles better suited to calculation than the observed angles, by adding to each of the latter, one-third of the preceding corrections 1″.15 and 1″.83. But if two of the observations should have been made more correctly than the third, then to this third is the correction chiefly to be applied. General Roy, probably, acted thus; for his corrections, as they now stand, appear rather arbitrary.

† This method has been used, in preference to the common method, because the logarithms of the two sides were already taken out. See page 83.

log. 79211.22 .... = 4.8987866
log. 71934.2 .... = 4.8569354

$\therefore$ log. tan. $\theta - 10$ .... = 0.0418512

$$\therefore \theta = 47° \; 45' \; 23''.1$$
$$\theta - 45° = 2 \;\; 45 \;\; 23.1$$

log. tan. $(\theta - 45°)$ .... = 8.6825575
cot. 26° 59′ 17″.69 .... = 10.2930544

$\therefore$ log. tan. $\left(\frac{A-B}{2}\right)$ .... = 8.9756119

$$\therefore \frac{A-B}{2} = 5° \; 24' \;\; 2''.325$$
$$\frac{A+B}{2} = 63 \;\;\; 0 \;\; 42.31$$
$$\therefore A = 68° \; 24' \; 44''.635$$
$$B = 57 \;\; 36 \;\; 39.985.$$

The Side c, opposite the included Angle, computed.

log. 71934.2 .... = 4.8569354
log. sin. 53° 58′ 35″.39 .... = 9.9078286

14.7647640

log. sin. 57° 36′ 39″.99 .... = 9.9265645

$\therefore$ log. c .... = 4.8381995

$$\therefore e = 68896.9.$$

We must now add to A and B one-third of the spherical excess, and we shall have the spherical triangle completely resolved; * that is,

3 Angles of the Spherical Triangle.	3 Sides of the Spherical Triangle.
$C' = 53° \; 58' \; 35''.75$	$c = 68896.9$
$A' = 68 \;\; 24 \;\; 44.995$	$a = 79211.22$
$B' = 57 \;\; 36 \;\; 40.345$	$b = 71934.2$
180 0 1.09	

---

* In this instance solved by this method of Legendre, the conditions of the triangle were, two sides and an included angle; but it is plain, that by the same method, a triangle with any three other conditions may be solved.

We shall have the same result as that which we have already obtained by resolving the Example according to the method employed in Case 3, page 166; but that method is rather long, and, on that account, it is superseded by the present, which, in cases like the preceding, gives results sufficiently exact. It must not, however, be forgotten, that, in fact, the preceding method is a method of approximation.

One or two cases may be brought under the conditions of the foregoing. If two sides of a spherical triangle are each nearly 180°, and the third side is very small, then, if we produce the two longer sides till they meet, a small spherical triangle will be formed, which may be resolved by the preceding method; and, from the sides and angles of this second triangle those of the first may be immediately deduced. Again, if two angles of a spherical triangle are very acute, the polar or supplemental triangle will have two sides nearly $=180^0$, and the third a very small side: this case, therefore, is under the conditions of the preceding one.

Triangles solved by the foregoing methods will be spherical triangles, their sides arcs of great circles, and the computed arc of the meridian also an arc of a great circle. But, the foregoing methods have not always been employed, and they are not indispensably necessary. Delambre, in France, and Colonel Mudge, in England, do not consider the line of the meridian as a curve, but as formed of the chords of curves; and they have resolved their triangles not as spherical triangles, either by the direct, or by Legendre's approximate process, but as rectilinear triangles, the sides of which are the chords of arcs. This method then requires Theorems and formulæ different from those that have been already explained. It, however, to a certain point, goes along with the former; for it supposes the oblique angles to be reduced, either instrumentally * or by calculation, (page 186) to horizontal

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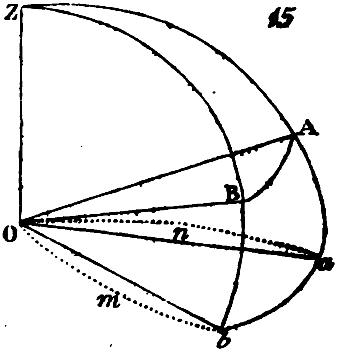
* Ramsden's Theodolite, by means of an azimuth circle, gives at once the horizontal angle when the oblique angle is observed; this instrument first, we believe, shewed the spherical excess, which rarely amounts to 3″. Delambre's and Mechain's Theodolites were Borda's repeating circles of small dimensions; with these the oblique angle was observed, and then reduced by calculation (see page 186.) to the horizontal angle.

angles; and, the horizontal angles being, as it has been already explained, spherical angles, must be reduced to the angles contained by the chords of the spherical sides of the triangle. To effect this reduction, we must resolve this Problem.

Given the angle contained by the spherical arcs *a n O*, *b m O*, (Fig. 15), the angle *a O b* contained by the chords *a O*, *b O* is required.



15

Let the spherical sides *a b*, *b m O*, *a n O*, be represented by *c*, *b*, *a*;

the chords *a b*, *b O*, *a O*..............γ, β, α;
the spherical angle at *O*................*C*
the angle *a O b* of the chord.............θ

then $\cos. c = \sin. a . \sin. b . \cos. C + \cos. a . \cos. b$ (page 139.).

But, $\cos. c = 1 - 2 \sin.^2 \frac{c}{2}$, and $\sin. a = 2 \sin. \frac{a}{2} \cos. \frac{a}{2}$, &c.

$$\therefore 1 - 2 . \sin.^2 \frac{c}{2} = 4 . \sin. \frac{a}{2} . \cos. \frac{a}{2} \sin. b . \cos. b . \cos. C +$$

$$\left\{1 - 2 \sin.^2 \frac{a}{2}\right\} \left\{1 - 2 \sin.^2 \frac{b}{2}\right\};$$

but, $\sin. \frac{c}{2}$, $\sin. \frac{a}{2}$, $\sin. \frac{b}{2}$, (see p. 10.) $= \frac{\gamma}{2}$, $\frac{\alpha}{2}$, $\frac{\beta}{2}$, respectively;

$\therefore$ substituting, transposing, &c.

$$2 \times \frac{\alpha^2+\beta^2-\gamma^2}{4} = \alpha\beta \cos. \frac{a}{2} \cos. \frac{b}{2} \cos. C + \frac{(\alpha\beta)^2}{4},$$

but (Prob. 1, pages 24, 25), $\cos. \theta = \frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta}$; ∴ substituting

$$\alpha\beta \times \cos. \theta = \alpha\beta \cos. \frac{a}{2} . \cos. \frac{b}{2} \cos. C + \frac{(\alpha\beta)^2}{4},$$

and dividing by $\alpha\beta$,

$$\cos. \theta = \cos. \frac{a}{2} . \cos. \frac{b}{2} . \cos. C + \sin. \frac{a}{2} . \sin. \frac{b}{2} .$$

From this expression, θ may be found by the method given in Case 3, page 171; thus, taking the instance given in page 187, the distance of Calais from Watten and Fiennes being in minutes and seconds of the Earth's meridian respectively 15′ 59″, 7′ 26″, we have $\frac{a}{2} = 7' \ 59''.5$, $\frac{b}{2} = 3' \ 43''$.

$$\therefore\ 2 \log. \cos. 33^\circ \ 15' \ 18''.44 = 19.8446590$$
$$\log. \cos. \ 0 \ \ 7 \ \ 59.5 = 9.9999988$$
$$\log. \cos. \ 0 \ \ 3 \ \ 43 = 9.9999998$$
$$\therefore\ 2 \log. \sin. M + 20 = 39.8446576$$
$$\text{and } M = 56^0 \ 44' \ 41''.$$

$$\text{Again, } 90^\circ - \frac{a+b}{4} = 89^\circ \ 54' \ 8.75;$$

$$\therefore\ 90^\circ - \frac{a+b}{4} + M = 180^\circ - (33^0 \ 21' \ 10''.25);\ \therefore\ \sin. = 9.7401996$$
$$90^\circ - \frac{a+b}{4} - M = \quad 33 \ \ 9 \ \ 27.75 \ldots\ldots \sin. = 9.7379440$$
$$2)19.4781436$$
$$\therefore\ \log. \sin. \frac{\theta}{2} = 9.7390718$$

$$\text{and } \theta = 66^\circ \ 30' \ 36''.68.$$

For reasons, however, similar to those which we used when speaking of the reduction of oblique angles to horizontal angles, the direct and ordinary rules of Spherical Trigonometry are too tedious for the occasion: θ differs very little from C, and a small correction only is required. To enable the computist to obtain such correction and with little trouble, M. Delambre, as in the former case of reduction, page 187, has investigated a formula, and constructed Tables resembling the former Tables.

In this method, then, M. Legendre's Theorem is not employed, nor is the Theorem relative to the spherical excess requisite; for there is this criterion of the accuracy of the observations—the sum of the three angles contained by the chords deduced from the spherical angles, ought, if the observed angles were truly observed, to equal 180°. The remainder of the calculation in this method is by the common rules of Plane Trigonometry; and the arc of the meridian, if that is proposed to be determined, is composed of the sides of an irregular polygon inscribed in a circle.*

We may then, in a Trigonometrical survey, calculate the arcs and angles according to the exact rules of Spherical Trigonometry, as Boscovich has done; or, we may reduce the observed angles to angles of the chords, and calculate, by Plane Trigonometry, such reduced angles and the chords, as Delambre and Mudge have done; or we may, by a slight transformation, give to spherical triangles the properties of plane triangles, and resolve them as such, according to Legendre's method.

Delambre, indeed, indefatigable in quest of accuracy, informs us, *Mem. Inst.* page 512, 1806, that he computed by the three different methods, the whole series of triangles that extend from Dunkirk to Barcelona.

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* The whole of the process of calculation in a Trigonometrical survey has not been described. Since the Earth is not a perfect sphere, some small correction is due to its spheroidical form; the investigation of these corrections is not of great difficulty, but is omitted here, since it would be foreign to the plan and matter of the present Treatise.

CHAP. XIII.

On the Relations between the corresponding Variations of the Angles and Sides and Triangles; and, on the Means of selecting, in the application of Trigonometrical Formulæ, the Conditions that are most favourable to accuracy of result.

THE preceding Chapter contains some illustration of the use of Trigonometrical formulæ. These formulæ are applied to certain data or conditions furnished by observation. Now, the Mathematical process is sure and infallible; but all instrumental observation, in a greater or less degree, is liable to error. The practical result then cannot be perfectly exact: but it will not, necessarily, be inexact to the full extent of the error of the observation. That error, according to the conditions of the case, will be variously modified by the Mathematical process. If it changes its magnitude by changing the conditions, it will be least when the conditions are of certain values. Hence, if it should happen, that we could vary the conditions, it would undoubtedly be expedient to assign them such a magnitude, that the errors of observations should least vitiate the results: that, in the words of a Mathematical statement, the error of the result should be least with a given error of observation.

These remarks stand, perhaps, in need of some illustration. The height of a tower may be determined by observing the angle which its summit makes with the horizon, and by measuring the horizontal distance between its base and the station of the observer. Now, in observing the angle, a certain error may be committed: but the error of the result (that which is Mathematically obtained) will vary as the distance between the tower and the observer is varied. If, therefore, we have it in our power, to regulate that condition, that is, if we can observe the height of the tower at what distance we please from its base, we plainly ought to select that which renders least inaccurate the result.

Again, in Astronomy, the time is determined from an observed altitude of the Sun or of a Star, from their declination and the latitude of the place. It is not then a question of mere curiosity to determine in what position, or part of the heavens, the Sun or Star ought to be observed, in order that the instrumental error, supposed to be of a certain magnitude, may least vitiate the determination of the time.

The determination of the least errors is only one branch of the general Problem, which assigns, in its solution, the relations between the corresponding errors in the data and results; that is, the given errors in one or more of the conditions of the Problem, and the *consequent* errors in the results. Thus, the right ascension and declination of the Sun are computed in the Nautical Almanack from the longitude furnished by the Solar Tables and from the obliquity of the ecliptic. Now, the determination of this latter condition is subject to some error. If we assign a value to that error, we may then investigate the corresponding errors in the right ascension and declination, and, in the result of such investigation, we should necessarily include the cases, in which the original error would least affect the values of the right ascension and declination.

The *errors*, that hitherto have been spoken of, are small variations or increments in the angles and sides of rectilinear and spherical triangles. Hence, an investigation of their corresponding values will comprehend a great variety of Problems that occur in Astronomy. For instance, it would assign the effects of parallax, refraction, aberration, precession, &c. in declination, right ascension, &c. since the effects of these inequalities, always very small, may be represented by very small portions of the arcs or circles along which those effects originally take place.

It is not here intended to extend this enquiry beyond triangles; but, there are a great variety of Problems belonging to other figures and other subjects of investigation that might have been included under the class of *Errores in mixtâ Mathesi*.

This was the Title which Roger Cotes gave to his Tract * on this subject; and Lacaille*, in treating of the same subject, properly describes the object of Cotes's Tract to be the *determination of the limits of inevitable errors in the practice of Geometry and Astronomy.*

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* Æstimatio errorum in mixta Mathesi per Variationes partium Trianguli plani et Spherici.

We purpose to treat this, as we have treated all the preceding subjects, analytically. Suppose the relation between an angle A and a side b, to be expressed by this equation,

$$\sin. A = m . \tan. b;$$

then, if A should be increased by ΔA, whilst b was increased by Δb (ΔA, Δb, representing the *entire* differences or increments of A and b), the equation belonging to the changed triangle would be

$$\sin. (A + \Delta A) = m . \tan. (b + \Delta b),$$

and the corresponding *errors* of A and b, or ΔA, Δb, would be to be determined from this equation, which is the difference of the two former, namely, from

$$\sin. (A + \Delta A) - \sin. A = m . \{ \tan. (b + \Delta b) - \tan. b \} .$$

If we expand † $\sin. (A + \Delta A)$, the left-hand side of the equation will become

$$\frac{d . \sin. A}{dA} . \Delta A + \frac{d^2 . \sin. A}{1 . 2 . dA^2} . (\Delta A)^2 + \&c.$$

Now, in most of the cases that come under this enquiry, ΔA, whether it represents the quantity of precession, or of parallax, or of aberration, &c. is always a very small quantity: so small, that without vitiating the result, we may reject all terms involving $(\Delta A)^2$; $(\Delta A)^3$, &c.; in which case, the preceding quantity would become

$$\frac{d . \sin. A}{d A} . \Delta A.$$

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* 'Le but de l'Auteur est de determiner les limites des erreurs inevitables dans la pratique de la Geometrie et de l'Astronomie.' *Acad. des Sciences*, 1741, p. 240.

† See *Principles of Analytical Calculation*, pp. 72, 73.

In like manner, if the right-hand side of the equation be evolved and the terms that involve $(\Delta b)^2$, $(\Delta b)^3$, &c. be rejected, it will be reduced to

$$\frac{d\,.\,\tan.\,b}{d\,b}\,.\,\Delta\,b.$$

Hence ΔA, $\Delta\,b$, are to be determined by this equation,

$$\frac{d\,.\,\sin.\,A}{d\,A}\,.\,\Delta\,A = m\,\frac{d\,.\,\tan.\,b}{d\,b}\,.\,\Delta\,b.$$

If $\Delta\,A$, Δb, should not be very small, or if considerable accuracy were required, the terms involving $(\Delta\,A)^2$, $(\Delta\,b)^2$ may be retained, in which case the equation will be

$$\frac{d\,.\sin.\,A}{d\,A}\,.\,\Delta\,A + \frac{d^2\,.\,\sin.\,A}{1\,.\,2\,.\,dA^2}\,.\,(\Delta\,A)^2 =$$

$$m\left\{\frac{d\,.\,\tan\,b}{d\,b}\,.\,\Delta\,b + \frac{d^2\,.\tan.\,b}{1\,.\,2\,.\,db^2}\,(\Delta\,b)^2\right\}$$

and for deducing $\Delta\,A$ in terms of $\Delta\,b$, or $\Delta\,b$ in terms of $\Delta\,A$; the solution of a quadratic would be requisite, $\frac{d\,.\sin.\,A}{d\,A}$, $\frac{d\,.\,\tan.\,b}{d\,b}$ (see *Principles of Anal. Calc.* p. 74,) are the differential coefficients of sin. A, tan. b, and are respectively equal to cos. A, sec.$^2\,b$.

We have taken a particular form; but, if we assume a general one, the method will be the same and the formula of solution similar. For instance, let X denote any function of A, and Y of b, and let the equation be

$$X = m\,Y;$$

then the equation for determining ΔA, Δb, will be

$$\frac{d\,X}{d\,A}\,.\,\Delta\,A + \frac{d^2\,X}{1\,.\,2\,.\,d\,A^2}\,.\,(\Delta\,A)^2 + \&c. =$$

$$m\left\{\frac{d\,Y}{d\,b}\,\Delta\,b + \frac{d^2\,Y}{1\,.\,2\,.\,d\,b^2}\,(\Delta\,b)^2,\ \&c.\right\}$$

and if $\Delta\,A$, $\Delta\,b$, are very small,

$$\frac{d\,X}{d\,A}\,.\,\Delta\,A = m\,\frac{d\,Y}{d\,b}\,.\,\Delta\,b,\text{ nearly};$$

$$\text{or, } \frac{dX}{dA} . \Delta A = \frac{X}{Y} \times \frac{dY}{db} . \Delta b, \text{ nearly.}$$

And, in like manner, if V should be a function of C, and U of a, &c. and the finite equation of relation should be

$$X + n . V + \&c. = m Y + p U + \&c.$$

n, m, &c. being constant quantities, the equation of relation between ΔA, Δa, &c. these quantities being very small, would be nearly

$$\frac{dX}{dA} . \Delta A + n . \frac{dV}{dC} . \Delta C = m . \frac{dY}{db} \Delta b + p . \frac{dU}{da} . \Delta a.$$

In order to facilitate the solutions of the following cases, we will prefix the values of the *differential coefficients* of sin. x, cos. x, &c.

$$* \frac{d . \sin. x}{dx} = \cos. x, \quad \frac{d . \cos. x}{dx} = -\sin. x, \quad \frac{d . \tan. x}{dx} = \sec.^2 x$$

$$\frac{d . \sec. x}{dx} = \frac{\tan. x}{\cos. x}, \quad \frac{d . \text{co-sec.} x}{d . x} = -\frac{\text{co-tan.} x}{\sin. x}, \quad \frac{d . \text{co-tan.} x}{dx} = -\text{co-sec.}^2 x.$$

EXAMPLE 1.

In a right-angled triangle, of which one side is h, the other a, and the angle opposite h, θ, it is required to find the *error* or variation in h, from a given error in θ, (See Cotes's *Est. Errorum in mixta Mathesi*, p. 20.)

$$\text{Here, } h = a . \tan. \theta ; \therefore \Delta h = a . \Delta \theta . \frac{d \tan. \theta}{d\theta} =$$

$$a . \Delta \theta . \sec.^2 \theta = \frac{\sec.^2 \theta}{\tan. \theta} . h \Delta \theta$$

$$= \frac{h . \Delta \theta}{\tan. \theta . \cos.^2 \theta} = \frac{h . \Delta \theta}{\sin. \theta . \cos. \theta} = \frac{2h . \Delta \theta}{\sin. 2\theta};$$

consequently, if $\Delta \theta$ be given, Δh will be least when sin. 2θ is the greatest, that is, when $\theta = 45°$, and consequently, when $a = h$.

---

* $\frac{d . \sin. x}{dx}$, in fluxionary notation is $\frac{\dot{\overline{\sin. x}}}{\dot{x}}$.

Hence, if h represent the height of a tower, and $\Delta\theta$ be the error of observation, (see p. 197, l. 22,) it will be most advantageous to observe the angular height of the tower at a distance about equal to its height *.

EXAMPLE 2.

In a right-angled spherical triangle, where C is the right angle, and A is invariable, it is required to find the corresponding variations of the hypothenuse c and the side b.

By Naper's first Rule, p. 146, making the complement of A the middle part, and the radius equal 1,

$$1 \times \cos. A = \tan. b \,.\, \cot. c;$$

∴ (see p. 201, l. 12,)

$$0 = \Delta b \cdot \sec.^2 b \,.\, \cot. c - \Delta c \,.\, \tan. b \,.\, \text{co-sec.}^2\, c;$$

$$\therefore \frac{\Delta b}{\Delta c} = \frac{\tan. b}{\sec.^2 b} \times \frac{\text{co-sec.}^2\, c}{\cot. c}$$

$$= \frac{\sin. b \,.\, \cos. b}{\sin. c \,.\, \cos. c} \text{ (see p. 9, 10.)}$$

$$= \frac{\sin. 2b}{\sin. 2c} \text{ (see p. 12.)}$$

EXAMPLE 3.

Let now c be invariable, and let it be required to find the ratio between the variations of the sides a, and b.

Make the complement of c the middle part, then, by Naper's second Rule, p. 146,

$$1 \times \cos. c = \cos. a \,.\, \cos. b;$$

∴ (p. 201, l. 12,)

$$0 = -\Delta a \cdot \sin. a \cdot \cos. b - \Delta b \,.\, \sin. b \cdot \cos. a;$$

---

* 'Commodissum erit ad eam distantiam (*AC*) observationem instituere ut angulus (*ACB*) sit graduum 45 quamproxime.' Cotes's *Est. Errorum*, p. 20.

$$\therefore \frac{\Delta a}{\Delta b} = -\frac{\text{sin.}\, b}{\text{cos.}\, b} \times \frac{\text{cos.}\, a}{\text{sin.}\, a} = -\text{ tan.}\, b \times \text{co-tan.}\, a.$$

Example 4.

In an oblique-angled spherical triangle (*SZP*), if one side (*PS*), vary, it is required to find the corresponding variation in one of the angles (*SPZ*). (See Lacaille, *Mem. Acad.* 1741, p. 242.)

Let the angles *SPZ*, *SZP*, *ZSP* be *A*, *C*, *B* respectively, and *a*, *c*, *b*, the opposite sides; then, see p. 139. of this Work,

$$\text{cos.}\, A\,.\,\text{sin.}\, b\,.\,\text{sin.}\, c = \text{cos.}\, a - \text{cos.}\, b\,.\,\text{cos.}\, c\,;$$

∴ by p. 201,

$$-\Delta A \text{ sin.}\, A\,.\,\text{sin.}\, b\,.\,\text{sin.}\, c + \Delta c\,.\,\text{cos.}\, c\,.\,\text{cos.}\, A\,.\,\text{sin.}\, b =$$

$$\Delta c\,.\,\text{cos.}\, b\,.\,\text{sin.}\, c\,;$$

$$\therefore \Delta A = \frac{\Delta c}{\text{sin.}\, A\,.\,\text{sin.}\, b\,.\,\text{sin.}\, c}(\text{cos.}\, A\,.\,\text{cos.}\, c\,.\,\text{sin.}\, b - \text{cos.}\, b\,.\,\text{sin.}\, c)$$

$$= \Delta c\,(\text{co-tan.}\, A\,.\,\text{cot.}\, c - \text{co-tan.}\, b\,.\,\text{cosec.}\, A).$$

If *Z* be the zenith, *P* the pole, and *S* the Sun, then the above solution will, in the method of finding the time by equal altitudes, assign the *correction* of the time (ΔA) which is due, by reason of the *variation* or *error* (Δc) in the co-declination. (See *Astron.* p. 404.)

Example 5.

In the preceding triangle (*SZP*) if *SZ* ($=a$) vary, it is required to find the corresponding variation in the angle *SPZ* ($= A$). (See *Estº. Errorum*, &c. p. 21.)

By p. 139. of this Work,

$$\text{cos.}\, A = \frac{\text{cos.}\, a - \text{cos.}\, b\,.\,\text{cos.}\, c}{\text{sin.}\, b\,.\,\text{sin.}\, c}\,;$$

∴ by p. 201,

$$-\Delta A\,.\,\text{sin.}\, A = -\Delta a\,.\,\frac{\text{sin.}\, a}{\text{sin.}\, b\,.\,\text{sin.}\, c}\,.$$

But, $\sin. A = \sin. C \times \frac{\sin. a}{\sin. c}$;

$$\therefore \frac{\Delta A}{\Delta a} = \frac{1}{\sin. b . \sin. C}.$$

Hence, since b is supposed to be constant, and C to be variable, ΔA is least, (Δa being given,) when sin. C is the greatest, that is, when C is a right angle.

If S, Z, &c. designate, what they were made to do, in the latter part of the preceding Example, then this solution determines the error in the time (ΔA) consequent on a given error in the observed altitude ($90° - a$), when from such altitude and the known latitude of the place, it is proposed to find the time; and, the solution also determines that the error in the time will be the least when $C (= SZP)$ is 90°, that is, when the star S is on the prime vertical. (See *Astron.* p. 408.)

By similar processes we might find (as Lacaille has done, *Mem. Acad.* 1741, p. 248.) the effects produced in the right ascensions and declinations of stars, by the precession of the equinoxes. But this, and like Problems require no new or peculiar principle for their solution, the first step of which (the essential one in this class of Problems) is to be made, as in the foregoing cases, by taking the differential or fluxion of each side of the equation (see p. 201, l. 12). The other steps necessary to produce results of a certain form, must vary with the conditions of the case, and consequently cannot be anticipated and prescribed by any fixed rules.

Here terminates what specially belongs to Trigonometry. In the course of the Treatise, considerable aid has been drawn from certain auxiliary branches of science: for instance, in almost every example, the processes and formulæ of logarithms have been introduced. Now, logarithms have neither a more intimate nor a more natural connexion with Trigonometry, than with many other branches of science. There is no eminent reason, then, why the properties of the former should be discussed in a Treatise on the latter science. Still, since it is usual to treat together of the one and the other, the custom is here not departed from. And, accordingly, for the purpose of investigating the properties of logarithms, and for the discussion of some other subjects connected with the preceding matter, the following Appendix is now added.

APPENDIX.

If we look to those branches of Science that are mathematically treated of, such as Dynamics, Astronomy, &c. we meet continually with instances, in which it is necessary to multiply numbers together, to divide one number by another, or to extract the roots of numbers. The common rules of arithmetic are adequate to these operations: but the operations themselves, especially if the numbers consist, as is generally the case, of several places of figures, are very tedious. The conditions of the problems that are really presented to us in Natural Philosophy for solution, are rarely expressed by small integer numbers. They are most frequently the results of methods of approximation; and, as such, are necessarily expressed either by decimals solely, or conjointly by integers and decimals. For example, according to the *method* of determining the eccentricity (e) of a planet's orbit (see *Astronomy*, vol. I. pp. 203, 204.) the eccentricity can never be *exactly* expressed: it is merely the result of a method of approximation: it can only, therefore, be nearly expressed: very nearly by five decimal places, more nearly by six, still more so by seven, and so on. The case is the same with other quantities: so that when we are obliged numerically to expand (which in practice we are always obliged to do) our formulæ, or the results of our mathematical processes, we have to multiply, divide, or extract the roots of such quantities as 1.016814, .983185, &c. ($1+e$, $1-e$); which operations, indeed, are not difficult, but are tedious; and, if of frequent recurrence, very embarrassing to the computist.

There is, besides, this circumstance to be noted in these simple operations of divisions, extractions, &c. namely, that the operations performed in any particular case cease, when the case is resolved, to be of farther use. The extractions of the square roots of an hundred numbers do not aid us in determining, with any increase of facility, the root of the hundredth and one number. Previous operations become not subservient to the abridgment of similar subsequent ones. The labours of preceding mathematicians are, in these cases, of no use to those that come after them.

These inconveniences, (such as have been described) could not but be felt by the early analysts: and, as it is natural, having once possessed themselves of sure methods of calculation, they began to seek after expeditious ones. After many trials and immense labour they discovered such, or rather invented such, by means of *Logarithms*.

These have had various definitions assigned to them, and have been computed by a great variety of methods. They have, with no great propriety of language, been styled *Artificial Numbers*. They have no more title to that denomination than the square or cube roots of the numbers 2, 3, 5, &c. have. If $10^x = 3$, x is the logarithm of 3, and is some number between 0 and 1, and must be expressed, for the practical purposes of computation, by some vulgar or decimal fraction. But $\sqrt{2}$, $\sqrt[3]{3}$, &c. are in the same predicament. There is no number that exactly expounds $\sqrt{2}$: its *value* (if we may so express ourselves) is between 1 and 2, but not capable of being exactly assigned: it is greater than $\frac{14}{10}$, but less than $\frac{3}{2}$; greater than $\frac{141}{100}$ but less than $\frac{142}{100}$, &c. &c.; and these *limits* between which the value of $\sqrt{2}$ is always placed, may be found either by the common rules for the extraction of roots, or by a series of tentative methods. The case is nearly the same with the equation $10^x = 3$. A series of limits between which x is, successively, still more and more narrowly placed, may be found by trial and the simplest operations: the value of x is between $\frac{1}{4}$ and $\frac{1}{2}$: it is less than $\frac{1}{2}$, greater than $\frac{3}{8}$: less than $\frac{1}{2}$, greater than $\frac{7}{16}$: less than $\frac{1}{2}$, greater than $\frac{15}{32}$; or less than .5, but greater than .46875: and, in this way, we may make approaches to the *logarithm* of 3, with as much certainty as towards the square or cube root of 2, 3, or of any other number which is not a complete power. The results are no more *artificial* in one case than in the other.

It is true that the direct process for approaching to the value of x in an equation such as $10^x = 3$, or $10^x = 2$, &c. is not so simple nor so easily practised as the ordinary processes or rules for the extraction of roots. But if we examine the matter on those grounds on which all analytical calculation rests, there is no essential difference between the two processes. They are in the same line of consecutive deductions: one nearer, indeed, to the common source than the other.

We have chosen to consider *logarithms* as the values of x in the equation $10^x = N$, when N is represented by the several numbers from 0 to 1000 and upwards. This, however, is not the form under which logarithms were originally exhibited, or need necessarily to be exhibited. It is the form rather to which (after many trials) as essentially embodying their properties, they have been reduced by analytic art. All numbers then are made equal to, or feigned to be equal to certain powers of 10, and the indices of those powers are the *logarithms* of the numbers. But it is plain if we assume such an equation as

$$10^x = N,$$

$$\text{or} \quad 10^{\log. N} = N.$$

We may also suppose $2^x = N$, or $3^x = N$, &c.; that is, there may be several *systems* of logarithms: alike in their general properties, but differing from each other by reason of their *bases*, which are the technical denominations of the numbers 2, 3, &c. in the equations $2^x = N$, $3^x = N$, &c.

The general formula for the value of x in the equation $N = a^x$ cannot be obtained by any ordinary or simple processes. But there are particular cases in which, without any trouble, we may assign the values of x: for instance, if 2 should be the *base*, then since

$$2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 16, \quad 2^5 = 32, \text{ \&c.}$$

2, 3, 4, 5, &c. would be the logarithms of 4, 8, 16, 32, &c. If 3 should be the *base*, then since

$$3^2 = 9, \quad 3^3 = 27, \quad 3^4 = 81, \text{ \&c.}$$

2, 3, 4, &c. in *such a system* would be the *logarithms* of 9, 27, 81, &c. In like manner, in the common or Briggs's System of Logarithms, in which 10 is the base,

2, 3, 4, 5, &c. are the logarithms of 100, 1000, 10000, 100000, &c.

There is no need of calculation, not even of the slightest, in these simple instances: and, if we selected others, we might still, by very simple, although tedious, processes, deduce or approximate to (and almost all the values are approximate ones) the values of the logarithms of numbers. For instance, if the *base* should be 2, then the logarithm of a number intermediate to $4 (= 2^2)$ and $8 (= 2^3)$ must be some number (using that term in in its general meaning) between 2 and 3: suppose the number to be $6 = 2^x$: then x is intermediate to 2 and 3: if the arithmetical mean of 2 and 3, namely $\frac{5}{2}$, be used to represent to, then, since $2^{\frac{5}{2}} = \sqrt{32}$ which is less than $\sqrt{36}\ (=6)$ $\frac{5}{2}$, or $2\frac{1}{2}$, or 2.5 is too small an index. If we assume $x = 2\frac{2}{3}$, or $\frac{8}{3}$, then since $2^{\frac{8}{3}} = \sqrt[3]{256}$ which is greater than $\sqrt[3]{216}\ (= 6)$, the index $2\frac{8}{3}$, or 2.26666, is too large. The logarithm of 6, therefore, is now confined within $2\frac{1}{2}$ and $2\frac{2}{3}$: if we try $\frac{31}{12} \left\{ = \frac{1}{2} \left(\frac{5}{2} + \frac{8}{3} \right) \right\}$ we shall find it too small: the next trial, therefore, must be with some number between the limits $\frac{31}{12}$ and $\frac{32}{12}$: and as these limits successively approach to each other in value, we shall, by repetition of trial, continually approach more and more nearly to the value of the logarithm of 6.

The above, however, is a very rude method of obtaining, and by trials not speedily made, the index of 2. It will, together with all similar imperfect and irregular methods, be superseded by any formula, or regular process, which, from the equation,

$$N = a^x,$$

exhibits x in terms of N and a.

From such a formula the arithmetical values of the logarithms might be deduced whatever were the system: whether it were

Briggs's, or the common system in which the *base* a is 10, or *Naper's*, or the *hyperbolic*, in which the *base* is 2.7182818, &c.

The use of such a formula, as that we have just spoken of, is to deduce the logarithms of numbers, principally of prime numbers. The use of logarithms is, as it has been stated, to render arithmetical operations more compendious *, than they are by the ordinary processes of multiplication, division, involution and evolution. That object (compendium of calculation) is attained partly by the registering of the arithmetical values of logarithms, when once obtained, and partly by the *properties* of logarithms. The arithmetical values of logarithms are obtained for us by the labours of others : and we should have indeed, on that score, the same kind of benefit, if the square, cube, &c. roots of numbers, the products of numbers, &c. obtained by previous computation, should be registered in Tables. But, as it is plain, the labour of computation would not only be excessive, but the size and number of the Tables would be so incommodious as to be almost entirely useless. The logarithms of numbers may be comprised within Tables of a convenient size : and may be applied, by means of the *properties of logarithms*, to all arithmetical operations whether of multiplication, division, evolution or involution. Their properties then (of which we shall now proceed to speak) render it worth the while (whatever the expence of time and labour) to procure the computation of several millions of results and to insert them in Tables.

Properties of Logarithms.

Let $N = a^{x}$, $N' = a^{x'}$, $N'' = a^{x''}$, $N''' = a^{x'''}$, &c. then $NN' = a^{x} \times a^{x'} = a^{x+x'}$; but by the definition, $x+x'$ is the logarithm of the number represented by NN', and x and x', by the same definition, are the logarithms of N and N'; hence we have, in other symbols,

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* The introduction to the English Translation of Briggs states, very plainly and distinctly, the uses of logarithms, "By them all troublesome multiplications and divisions in arithmetic are avoided, and performed only by addition instead of multiplication, and by subtraction instead of division. The curious and laborious extractions of roots are also performed with great ease, as the square root of any number is found by bipartition or division by 2, &c." *Logarithmetical Arithmetic*, 1631.

$$\log.\,(N\,N') = \log.\,N + \log.\,N'.$$

If therefore we possess already computed the logarithms of numbers, instead of multiplying, when there is occasion, one number by another, it is sufficient simply to add their logarithms; and, the sum will be a logarithm corresponding to which, is the number that is the product required. For instance,

the logarithm of 2.13 .... = 0.3283796
of 47.2 .... = 1.6739420
2.0023216

and, the number corresponding to 2.0023216 is 100.536, which is the product of 2.13, and 47.2; as on trial, it will be found to be.

Again, $NN'N'' = a^x \times a^{x'} \times a^{x''} = a^{x+x'+x''}$; but by the definition, $x + x' + x''$ is the logarithm of $NN'N''$, or,

$$\log.\,N + \log.\,N' + \log.\,N'' = \log.\,(NN'N'').$$

Again, $\frac{N}{N'} = \frac{a^x}{a^{x'}} = a^{x-x'}$, but by the definition, $x - x'$ is the logarithm of the number corresponding to $a^{x-x'}$, or $\frac{N}{N'}$: or,

$$\log.\,N - \log.\,N' = \log.\,\frac{N}{N'};$$

hence, instead of dividing, for instance, 841.32 by 5.316, subtract the logarithm of 5.316 from that of 841.32, and the remainder is a logarithm, corresponding to which is a number that is the quotient that would arise on dividing 841.32 by 5.316;

Again, $N^m = (a^x)^m = a^{mx}$, but, by the definition, mx is the logarithm of a^{mx}, or N^m; ∴ in other symbols,

$$m \log.\,N = \log.\,N^m;$$

hence, instead, for instance, of multiplying 53.127, five times by itself, in order to obtain $(53.127)^6$, multiply the logarithm of 53.127 by 6, and the product is a logarithm, the number corresponding to which is the 6th power of 53.127.

Again, $N^{\frac{1}{n}} = (a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}$: but by the definition, $\frac{x}{n}$ is the logarithm of a number represented by $a^{\frac{x}{n}}$, or $N^{\frac{1}{n}}$: or in other symbols,

$$\frac{1}{n}\,.\log.\,N = \log.\,N^{\frac{1}{n}}\,.$$

Hence, instead, for instance, of extracting the square root of 137.51 twice, in order to obtain $\sqrt[4]{(137.51)}$, divide the logarithm of 137.51 by 4, and the result is a logarithm: and the number corresponding to it is the fourth or Biquadratic Root of 137.51.

Even these few illustrations shew the utility of logarithms: by means of a few simple rules and the same Tables, or registered results, the complex operations of arithmetic (as they may be called) or the involution and evolution of numbers, are superseded by the most simple, which are addition or subtraction. The enquiry, therefore, may be now directed towards what, indeed, are, in this subject, the sole remaining objects of curiosity, the certain methods of computing logarithms from numbers, and numbers from logarithms. These objects will, it is evident, be attained by the solutions of two problems, by one of which, the equation being

$$N = a^x,$$

x should be expressed in terms of N and a: and by the other N should be expressed in terms of a and x; the expressions in each case admitting (which is essential) of an easy application in specific instances.

Expansion of a^x.

$$a^x = \{1 + (a - 1)\}^x$$

$$= 1 + x\,.(a - 1) + x\,.\frac{x - 1}{2}\,.(a - 1)^2$$

$+ \; x \, . \, \frac{x-1}{2} \, . \, \frac{x-2}{3} \, . \, (a-1)^3 + \&c.$ by the Binomial Theorem*: and arranging the series by the powers of x; $a^x = 1 + x \{ (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c. \} + q x^2 + r x^3 + \&c.$ q, r, &c. standing for certain combinations of $(a-1)^2, (a-1)^3$, with the numbers, 2, 3, &c.: for, it is plain, from the manner by which they must be formed, that they cannot involve the index x, or any function of it:

hence, if we put $p = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$

$$a^x = 1 + p x + q x^2 + r x^3 + \&c.$$

$$\text{and } a^z = 1 + p z + q z^2 + r z^3 + \&c.$$

$\therefore a^x \times a^z$, or $a^{x+z} = 1 + p(x+z) + p^2 x z + q(x^2 + z^2) + p q (x^2 z + z^2 x) + \&c.$; but from the original form for a^x, substituting $x + z$ instead of x, we have

$$a^{x+z} = 1 + p(x+z) + q(x+z)^2 + r(x+z)^3 + \&c.$$

$$= 1 + p(x+z) + q \, . \, 2 x z + q(x^2 + z^2) + r(3 x^2 z + 3 x z^2) + \&c.$$

hence, comparing the terms that involve like powers and combinations of x and z;

$$p = p,$$

$$2q = p^2, \text{ and } \therefore q = \frac{p^2}{2},$$

$$3r = pq, \text{ and } \therefore r = \frac{pq}{3} = \frac{p^3}{2\,.\,3};$$

and, if s represent the coefficient of the term succeeding $r x^3$,

$$4s = rp, \text{ and } \therefore s = \frac{rp}{4} = \frac{p^4}{2\,.\,3\,.\,4};$$

$$\therefore a^x, \text{ or } N = 1 + p x + \frac{p^2}{1.2} \, . \, x^2 + \frac{p^3}{1.2.3} \, . \, x^3 + \frac{p^4}{1\,.\,2\,.\,3\,.\,4} \, x^4 + \&c. \quad [n]$$

This series then determines the number N in terms of the base a, and the index or logarithm x; but, the reverse Problem, or that which determines x in the terms of N and a is of the most consequence.

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* See Woodhouse's *Principles of Analytical Calculation*, p. 35, &c.

x *expressed in terms of* N *and* a.

$$\text{Since } a^x = 1 + px + \frac{p^2}{1.2}x^2 + \frac{p^3}{1.2.3}x^3 + \&\text{c.}$$

$$N^z = 1 + Pz + \frac{P^2}{1.2}z^2 + \frac{P^3}{1.2.3}z^3 + \&\text{c.}$$

$$\text{if } P = (N-1) - \tfrac{1}{2}(N-1)^2 + \tfrac{1}{3}(N-1)^3 - \&\text{c.}$$

but, since $N = a^x$, $N^z = a^{xz} = 1 + pxz + \frac{(px)^2}{1.2}.z^2 + \&\text{c.}$

Hence, comparing the terms that are affected with like powers of z, $px = P$, $\frac{(px)^2}{1.2} = \frac{P^2}{1.2}$, the same equation in fact as $px = P$; hence $x = \frac{P}{p} = \frac{(N-1) - \frac{1}{2}.(N-1)^2 + \frac{1}{3}(N-1)^3 - \&\text{c.}}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&\text{c.}}$ [*l*]

and hence, if instead of $N = a^x$, the equation is $1 \pm N = a^x$, we shall have $x = \frac{\pm N - \frac{1}{2}N^2 \pm \frac{1}{3}N^3 - \&\text{c.}}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&\text{c.}}$ [*L*].

This is a simple algebraical mode of expressing x: but, it does not follow that it is, in all instances, commodious for the arithmetical computation of x; since, if N be represented by any number, 7 for instance, and a be 10, the base of the common logarithms, neither the series $6 - \frac{1}{2}(36) + \frac{1}{3}(216) - \&\text{c.}$ representing the numerator, nor the series $9 - \frac{1}{2}.(81) + \frac{1}{3}(729) - \&\text{c.}$ representing the denominator, converge: in fact, the terms of the series are larger, the more remote they are from the beginning, and consequently no number of them summed can exhibit, either exactly or nearly, the true sum. Retaining then the law of the expressions, we must now adapt them to numerical computation: and first we will shew a method of computing

$$p = (a-1) - \tfrac{1}{2}(a-1)^2 + \&\text{c.}$$

In the series $a^x = 1 + px + \frac{p^2}{1.2}x^2 + \frac{p^3}{1.2.3}x^3 + \&c.$ let $x = 1$, and $p = 1$, then a will have a peculiar value, and be $= 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c. = 2.7182818284$, &c. call this e, then $(e - 1) - \frac{1}{2}(e - 1)^2 + \frac{1}{3}(e - 1)^3 - \&c.$ (which is the value of p in this case) $= 1$.

In the expression for a^x put $x = \frac{1}{p}$, then

$$a^{\frac{1}{p}} = 1 + 1 + \frac{1}{1.2} + \&c. = e, \text{ and } \frac{1}{a} = e^{-p};$$

but if $e^{-p} = \frac{1}{a}$, then, by the form [l], page 214,

$$-p = \frac{\left(\frac{1}{a} - 1\right) - \left(\frac{1}{a} - 1\right)^2 + \frac{1}{3}\left(\frac{1}{a} - 1\right)^3 - \&c.}{(e - 1) - \frac{1}{2}(e - 1)^2 + \frac{1}{3}(e - 1)^3 - \&c.}$$

or, since the denominator $= 1$,

$$p = \frac{a - 1}{a} + \frac{1}{2}\left(\frac{a - 1}{a}\right)^2 + \frac{1}{3}\left(\frac{a - 1}{a}\right)^3 + \&c.$$

Now, since $a - 1 < a$, this series always converges. Hence, in the common system, or Briggs's, in which $a = 10$,

$$p = .9 + \frac{(.9)^2}{2} + \frac{(.9)^2}{3} + \&c. = 2.3025850929, \&c.$$

Let us now endeavour to obtain the numerator by means of a converging series.

By the form [L], page 214, it appears, that

$$\text{if } 1 + n = a^x,\ x = \frac{1}{p}\left(n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \&c.\right)$$

$$\text{and, if } 1 - n = a^y,\ y = \frac{1}{p}\left(-n - \frac{1}{2}n^2 - \frac{1}{3}n^3 - \&c.\right)$$

Hence,

$$\frac{1+n}{1-n}=a^{x-y}, \text{ and, } x-y, =\frac{2}{p}\left(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\&c.\right)$$

$$\text{or, log.}\left(\frac{1+n}{1-n}\right)=\frac{2}{p}\left(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\&c.\right)$$

Now, $N=\dfrac{1+\dfrac{N-1}{N+1}}{1-\dfrac{N-1}{N+1}}$, substitute therefore, in the preceding form, $\dfrac{N-1}{N+1}$ instead of n, and we shall have

$$\text{log. } N=\frac{2}{p}\left\{\frac{N-1}{N+1}+\tfrac{1}{3}\left(\frac{N-1}{N+1}\right)^3+\tfrac{1}{5}\left(\frac{N-1}{N+1}\right)^5+\&c.\right\} \quad [\lambda]$$

and this series is plainly a converging series, for $\dfrac{N-1}{N+1}$ being a proper fraction, the terms, reckoning from the beginning of the series, are less and less.

For other methods, see the *Principles of Analytical Calculation*, pp. 137, &c.

Example.

Let it be required to compute the logarithms of 2 and 3.

$N = 2$, and $\frac{N-1}{N+1} = \frac{1}{3}$

$$\frac{1}{3} \ldots\ldots\ldots\ldots = .33333333$$

$$\frac{1}{3 \cdot 3^3} \ldots\ldots\ldots\ldots = .01234567$$

$$\frac{1}{5 \cdot 3^5} \ldots\ldots\ldots\ldots = .00082304$$

$$\frac{1}{7 \cdot 3^7} \ldots\ldots\ldots\ldots = .00006532$$

$$\frac{1}{9 \cdot 3^9} \ldots\ldots\ldots\ldots = .00000564$$

$$\frac{1}{11 \cdot 3^{11}} \ldots\ldots\ldots\ldots = .00000051$$

$$\frac{1}{13 \cdot 3^{13}} \ldots\ldots\ldots\ldots = .00000004$$

$$\overline{.34657355}$$

$\therefore$ log. 2 =

$$\frac{2}{2.30258509 \text{ \&c.}} \times .3465735$$

= .3010300 to 7 places.

$N = 3$, and $\frac{N-1}{N+1} = \frac{2}{4} = \frac{1}{2}$

$$\frac{1}{2} \ldots\ldots\ldots\ldots = .5$$

$$\frac{1}{3 \cdot 2^3} \ldots\ldots\ldots\ldots = .0416666666$$

$$\frac{1}{5 \cdot 2^5} \ldots\ldots\ldots\ldots = .00625$$

$$\frac{1}{7 \cdot 2^7} \ldots\ldots\ldots\ldots = .0011160714$$

$$\frac{1}{9 \cdot 2^9} \ldots\ldots\ldots\ldots = .0002170138$$

$$\frac{1}{11 \cdot 2^{11}} \ldots\ldots\ldots\ldots = .0000443892$$

$$\frac{1}{13 \cdot 2^{13}} \ldots\ldots\ldots\ldots = .0000093900$$

$$\frac{1}{15 \cdot 2^{15}} \ldots\ldots\ldots\ldots = .0000020345$$

$$\frac{1}{17 \cdot 2^{17}} \ldots\ldots\ldots\ldots = .0000004487$$

$$\frac{1}{19 \cdot 2^{19}} \ldots\ldots\ldots\ldots = .0000001003$$

$$\frac{1}{21 \cdot 2^{21}} \ldots\ldots\ldots\ldots = .0000000227$$

$$\frac{1}{23 \cdot 2^{23}} \ldots\ldots\ldots\ldots = .0000000050$$

$$\overline{.5493061422}$$

$\therefore$ log. 3 =

$$\frac{2}{2.30258509}, \text{\&c.} \times .5493061$$

= .4771212.

The logarithms of 2 and 3 thus computed, are the logarithms in Briggs's, or the common system, in which $a = 10$ and $(a - 1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}.(a-1)^3 -$ &c. $= 2.30258509$, &c.

If we take the base $= e = 2.71828182$, &c., in which $(e-1) - \frac{1}{2}.(e-1)^2 + \frac{1}{3}(e-1)^3 -$&c.$=1$, the logarithms will belong to a system called, from certain analogies, *Hyperbolic*, or from its inventor, *Naper's*; and accordingly, we shall have the hyperbolic logarithm of numbers from the preceding series, by omitting the denominator p which is equal to 1,

since it $= (e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 -$ &c.

hence, hyp. log. 2 = .69314718, and hyp. log. 3 = 1.098612, &c.

A constant multiplier connects one system of logarithms with another: if h be the hyperbolic logarithm of a number N, then, $h \times \frac{1}{2.30258509, \&c.}$, or $h \times .43429448$, &c. is the common logarithm. Generally, if the base of a system of logarithms be b, then, in that system, the logarithm of a number $N = \frac{h}{B}$,

if $B = (b-1) - \frac{1}{2}.(b-1)^2 + \frac{1}{3}(b-1)^3 -$&c.

or, if x be the logarithm of a number N in a system of which the base is a, then log. N (base b) $=$

$$x.\frac{p}{B} = x \times \frac{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}{(b-1) - \frac{1}{2}(b-1)^2 + \frac{1}{3}(b-1)^3 - \&c.}.$$

By means of the logarithmic series, page 214, the logarithms of 2 and 3 have been computed; and, by the aid of the same series, by the properties of logarithms, and by certain simple decompositions of numbers, the logarithms of all other numbers may be found: for instance,

The logarithms of 5 *, 7, 11 may be computed from the series.

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* Since $5 = \frac{10}{2}$, log. $5 = \log. \frac{10}{2} = \log. 10 - \log. 2 = 1 - \log. 2 = 1 - .3010300 = .6989700$; ∴ it is unnecessary to compute the log. 5 by the series.

The logarithms of 4, 6, 8, 9, 10, 12, 14, 15, 16, 64, &c. will follow from the properties of logarithms, for

log. $4 =$ log. 2^2 .... $= 2$ log. 2
log. $6 =$ log. (2×3) .... $=$ log. $2 +$ log. 3.
log. $8 =$ log. 2^3 .... $= 3$ log. 2
log. $9 =$ log. 3^2 .... $= 2$ log. 3
log. $10 =$ log. (2×5) .... $=$ log. $2 +$ log. 5
log. $12 =$ log. $3 +$ log. 4 .... $=$ log. $3 + 2$ log. 2
log. $14 =$ .... $=$ log. $2 +$ log. 7
log. $15 =$ .... $=$ log. $3 +$ log. 5
log. $16 =$ log. 2^4 .... $= 4$ log. 2
log. $64 =$ log. 2^6 .... $= 6$ log. 2.

The logarithms of 13, 17, 19, 23, 29, &c. cannot be easily computed from the series [λ], since $\frac{N-1}{N+1}$, the larger N is, approaches to 1: with the preceding numbers 13, 17, &c. the fraction would be respectively, $\frac{12}{14}$, or $\frac{6}{7}$, $\frac{16}{18}$, or $\frac{8}{9}$, $\frac{11}{12}$, $\frac{14}{15}$, &c. and the numbers also being prime cannot be resolved into factors; but if they algebraically be expressed after this manner, viz.

$$N = (N-1)\left(1 + \frac{1}{N-1}\right), \text{ then}$$

$$\text{log. } N = \text{log. } (N-1) + \text{log. } \left(1 + \frac{1}{N-1}\right) =$$

$$\text{log. } (N-1) + \frac{1}{p}\left(\frac{1}{N-1} - \frac{1}{2}\frac{1}{(N-1)^2} + \frac{1}{3}\frac{1}{(N-1)^3} - \&c.\right)$$

by [L], page 214; and thus, the logarithms may be computed by series that converge with sufficient rapidity.

For instance,

$$\text{if } N=13; \text{ log. } 13 = \text{log. } 12 + \frac{1}{p}\left(\frac{1}{12} - \frac{1}{2.12^2} + \frac{1}{3.12^3} - \&c.\right)$$

$$\text{if } N=17; \text{ log. } 17 = \text{log. } 16 + \frac{1}{p}\left(\frac{1}{16} - \frac{1}{2.16^2} + \frac{1}{3.16^3} - \&c.\right)$$

$$\text{if } N=23; \text{ log. } 23 = \text{log. } 22 + \frac{1}{p}\left(\frac{1}{22} - \frac{1}{2.22^2} + \frac{1}{3.22^3} - \&c.\right)$$

$$\text{if } N=29; \text{ log. } 29 = \text{log. } 28 + \frac{1}{p}\left(\frac{1}{28} - \frac{1}{2.28^2} + \frac{1}{3.28^3} - \&c.\right)$$

In these expressions, the logarithms of 12, 16, 22, 28, are known from the logarithms of their factors, see p. 219: and when N is a prime number, $N - 1$ can be aways resolved into factors. There are, however, besides the preceding, various other artifices and methods for computing logarithms *.

But, as it has been in substance remarked before, the art of computing logarithms, and dexterity in that art, would, by themselves, be of no use in expediting calculation: if, for instance, we had to multiply 31.523 by 17.81, and to divide the product by 5.4312, it would be a most long method of performing the operation, to investigate the logarithms of these numbers. It is the circumstance of registering computed logarithms in Tables, and, by the art of printing, of multiplying such Tables, that enables us to compute quickly. The calculation of logarithms is exceedingly operose; but one man calculates for thousands, and the results of tedious operations are made subservient to the abridgment of similar ones.

By the methods already described, the logarithms of all numbers from 1 to 100000, are computed and registered in Tables. Those in common use contain the logarithms of numbers, according to Briggs's System, in which the base (a) is 10. Naper's, or the Hyperbolic logarithms are so seldom required in numerical calculation, that it is more convenient to deduce them from Briggs's, by multiplying the latter into the number 2.30258509299, &c. than to search for them in separate Tables †.

But Naper's System, in which, $(e-1)-\frac{1}{2}(e-1)^2+\frac{1}{3}(e-1)^3 - \&c. = 1$ ($e =$ base) is, apparently, so very simple, that there must exist some substantial reason for the adoption of Briggs's. Now, in this latter system, the logarithm of 10 is 1, the logarithms of 100, or 10^2, of 1000, or 10^3, &c. are 2, 3, &c. respectively; consequently, the logarithm (L) of a number N being

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* See *Principles of Anal. Calc.* pages 142 to 183: *Phil. Trans.* 1806, p. 327: Bertrand, p. 421 to 676.

† Thomas Simpson, has given a short Table of Hyperbolic Logarithms at the end of his Fluxions: in Callet's and Hutton's Logarithms there is a Table, of a single page, for converting common into Hyperbolic Logarithms.

known, the logarithms of all numbers corresponding to $N \times 10^m$, or $\frac{N}{10^m}$ can be expressed by an alteration in L of the simplest kind. Thus, if the logarithm of 2.7341 be .4368144, the logarithms of the numbers 27.341, 273.41, 2734.1, 27341, 273410, are 1.4368144, 2.4368144, 3.4368144, 4.4368144, 5.4368144; that is, these latter logarithms are formed from the first by merely prefixing to the decimal, 1, 2, 3, 4, 5, which are called *characteristics*, and which characteristics are always numbers one less than the number of the figures of the integers in the numbers whose logarithms are required: the reason is this,

$$27.341 = 10 \times 2.7341 \,;\ \therefore \text{log.}\, 27.341 = \text{log.}\, 10 + \text{log.}\, 2.7341 = 1.4368144$$
$$2734.1 = 1000 \times 2.7341 \,;\ \therefore \text{log.}\, 2734.1 = \text{log.}\, 1000 + \text{log.}\, 2.7341 = 3.4368144$$

and generally, $\text{log.}\, 10^m \times N = \text{log.}\, 10^m + \text{log.}\, N = m + L$ and, similarly, it is plain, that the logarithms of

$$\frac{2.7341}{10},\ \frac{2.7341}{100},\ \frac{2.7341}{1000},\ \frac{2.7341}{10{,}000},\ \text{that is, of}$$
$$.27341,\quad .027341,\quad .0027341,\quad .00027341,$$

must be the logarithm of 2.7341, or .4368144, subtracting, respectively, the numbers 1, 2, 3, 4, which subtraction, it is usual thus to indicate:

$$\bar{1}.4368144,\qquad \bar{2}.4368144,\qquad \bar{3}.4368144,\qquad \bar{4}.4368144$$

The logarithm of a number (N), then, being inserted in the Tables, it is needless to insert the logarithms of those numbers that can be formed by multiplying or dividing N by 10 and powers of 10.

Hence, we are enabled to contract the size of logarithmic Tables: and this advantage is peculiarly connected with the decimal system of notation. If there had been, in common use, scales of notation, the *roots* * of which were 9, or 7, or 3: then the most *convenient* systems of logarithms would have been those, the bases of which, are 9, 7, 3, respectively. For, in such cases, after having computed the logarithm of any number N, we could immediately, by means of the characteristics, assign the logarithm of any number represented by $9^m \times N$, or $\frac{N}{9^m}$ [root = 9,] which numbers would, analogously to the present method, be denoted by merely altering the place of the point or comma that separates integers from fractions. The *root*, then, in the scale of notation ought to determine the choice of the base in a system of logarithms. We may construct logarithms with a base = 3, and then, having computed the logarithm L of a number N, the logarithms of all numbers corresponding to $3^m \times N$, and $\frac{N}{3^m}$ would be $m + L$, and $-m + L$, and, therefore, could be assigned by merely prefixing the proper characteristics; but then, in order to know the numbers corresponding to $3^m \times N$ and $\frac{N}{3^m}$, we must multiply and divide N by 3, and the powers of 3. We cannot multiply and divide by simply altering the place of the point or comma that separates integers from decimals; so that, in fact, not knowing, by inspection, such numbers as $3\,N$, $9\,N$, $27\,N$, and $3^m \times N$, we should be obliged to insert the logarithms of *all* numbers in the Tables.

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* The *root* or *radix* of a scale is that number according to the powers of which any digit, as it is moved more and more towards the left, increases in value: in our scale the root is 10: thus $723 = 7 \times 10^2 + 2 \times 10 + 3$.

A single instance will elucidate this statement: with a base =3, the logarithm of 2.7341 equals .915519, then the logarithms of the numbers

8.2023, 24.6069, 73.8207, 221.4621
are 1.915529, 2.915519, 3.915519, 4.915519;

for, the numbers 8.2023, 24.6069, &c. are produced by multiplying 2.7341 by 3, 3^2, 3^3, 3^4, respectively; they are known, however, only by actual multiplication, and consequently it would not be sufficient to insert in Tables of logarithms constructed to a base = 3, the logarithm of 2.7341 only; but, those of 8.2023, 24.6069, 73.8207, 221.4621, &c. must be also inserted: and it is plain, that the logarithms of 27.341, 273.41, 2734.1, 27341, .027341 must be also inserted. If these latter logarithms are not inserted, the computist would be obliged to undergo the labour of forming them, by adding to the logarithm of 2.7341, respectively, the logarithms of 10, 100, 1000, &c. computed to a base = 3.

This is not the sole principal inconvenience that would arise from using a system of logarithms with a base not equal to 10. We might indeed, as it has been explained, by slight Arithmetical operations, directly find the logarithms of numbers from Tables of no greater extent than those which are in use; but, the reverse operation of finding the number from the logarithm, could not at all conveniently or briefly be performed: for, the logarithm proposed might be nearly equal to a logarithm which the Tables did not contain. These considerations will, perhaps, be sufficient to shew the very great improvement that necessarily ensued on Briggs's alteration of the logarithmic base. The real value of that alteration does not seem to have been duly appreciated by writers on this subject.

For the description and use of Tables, in which the computed logarithms of numbers are recorded, the Reader is referred to the volumes of the Tables themselves: and, as nothing seems wanting to the plainness and precision of the rules therein delivered, it would be a needless accumulation of matter to insert them here. The principle however of the construction of certain small Tables for proportional parts, that are nearest the margin of every page, requires explanation. The use of these Tables is to find the logarithms of numbers consisting of more than five places. See Sherwin, p. 6, Hutton, p. 128, first edition.

Let the number composed of the first five figures or digits of the number N be n; therefore, the number next to n, or which differs from n by 1, is $n+1$; let x be the digit, which placed after the digits composing n, shall make it N, then $N=10n+x$, and

$$\log. N = \log. (10n+x) = \log. 10n\left(1+\frac{x}{10n}\right),$$

$$= \log. 10n + \log. \left(1+\frac{x}{10n}\right) \text{ [p. 211]},$$

$$= \log. 10n + \frac{1}{p}\left\{\frac{x}{10n} - \frac{1}{2}\frac{x^2}{(10n)^2} + \&c.\right\}$$

$= \log. 10n + \frac{x}{p \times 10n}$, if the terms $\frac{x^2}{2 . (10n)^2}$, &c. are, on account of their minuteness, neglected.

Again, $\log. \frac{n+1}{n} = \log. \left(1 + \frac{1}{n}\right) = \frac{1}{p} . \frac{1}{n}$ [neglecting $\frac{1}{2p . n^2}$, &c.] consequently, $\frac{1}{pn} = \log. (n + 1) - \log. n$, and

$$\frac{x}{p \times 10n} = \frac{x}{10} \{ \log. (n + 1 - \log. n) \},$$

and from this formula the small Tables of the proportional parts may be computed: for instance,

Let $N = 678323$, then $n = 67832$, and $n + 1 = 67833$, and $\log. 67833 - \log. 67832 = 8314410 - 8314346 = 64$, and since $x = 3$, $\frac{x}{10} \{ \log. (n + 1) - \log. n \} = \frac{3}{10} \times 64 = 19.2$, (or taking the nearest whole number) $= 19$: and by putting for x, 1, 2, 3, &c. we may form the small Table which is in the page containing the number 6783, &c. thus:

	Proportional Part.	Proportional Part in the nearest integers.
$x = 1$	6.4	6
2	12.8	13
3	19.2	19
4	25.6	26
5	32	32
6	38.4	38
7	44.8	45
8	51.2	51
9	57.6	58

See Sherwin's Tables, page 6, and at number 6783, and also Hutton's, page 128.

The above proof establishes the truth of the precept for finding the logarithms of numbers consisting of more than 5 places of figures: the other precept * which directs us to find the number corresponding to a logarithm not found exactly in Tables, may be thus proved.

Let L be the proposed logarithm, N the number: l the tabular logarithm next less; l' the tabular logarithm next greater; n, n', their corresponding numbers.

Let x be the difference of N and n, or let $N = n + x$; then $\log. N = \log. (n + x) = \log. n\left(1 + \frac{x}{n}\right) = \log. n + \log. \left(1 + \frac{x}{n}\right)$

$$\text{or, } L = l + \frac{x}{np}, \text{ nearly; } \therefore x = np\,(L - l).$$

Again,

$\log. n' = \log. (n + 1) = \log. n\left(1 + \frac{1}{n}\right) = \log. n + \log. \left(1 + \frac{1}{n}\right)$

or $l' = l + \frac{1}{np}$; $\therefore l' - l = \frac{1}{np}$, consequently $x = \frac{L - l}{l' - l}$, and $N = n + x = n + \frac{L - l}{l' - l}$; from which expression, the precept (Sherwin, p. 8. Hutton, p. 130.) and the small Table are derived: for instance, let $L = .4414728$, then (see the Tables),

$$l = .4414595, \quad n = 27635$$
$$l' = .4414752, \quad n' = 27636$$

$$\therefore L - l = 133, \; l' = 157, \text{ and } \frac{133}{157} = \frac{1}{10} \times \frac{1330}{157} =$$

$$\frac{1}{10}\,\frac{1256 + 74}{157} = .8 + \frac{74}{1570} = .8 + \frac{1}{100} \times \frac{7400}{1570} = .8 +$$

$\frac{1}{100} \times \frac{6280 + 1120}{1570} = .8 + .04 + \&c. = .84 + \&c.$ 8 and 4 being the two figures given according to the Rule and Table, 8 corresponding to 125.6, or 126 the nearest integer, and 4 to 62.8, or 63 the nearest integer.

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* Hutton, p. 130, first edition: Sherwin, p. 8, fifth edition.

It may now be worth the while to illustrate, by a few more instances, the uses of logarithms; and this will be done chiefly with a view of relieving the Student from any embarrassment which the *negative index* or *characteristic* (see p. 221.) as it is called, may occasion.

In the common system of logarithms in which the *base* is 10, 1 is the logarithm of 10, and 0 the logarithm of 1: consequently, every number than can be assigned between 10 and 1 must have for its logarithm a proper fraction, or (since a fraction may always be decimally expressed) a decimal fraction. The logarithms, therefore, of 2, 3, 4.56, 6.9345, &c. must be such decimals as

.3010300, .47712125, .6589648, .8410152, &c.

which, as it has been already argued (see pp. 207, &c.) are not to be called *artificial* numbers, but are computed numbers such as make good the equations,

$$10^{.3010300} = 2 \quad , \quad 10^{.47712125} = 3$$
$$10^{.6589648} = 4.56, \quad 10^{.8410152} = 6.9345.$$

The Logarithmic Tables contain, in fact, the logarithms only of those numbers which are contained between 1 and 10; and, from these registered logarithms, those of other numbers, less than 1, and greater than 10, are to be derived by means of the properties of logarithms. Thus, in the following extract from Sherwin's Tables:

Numbers.	Logarithms.	
1255	093	6437
56		9896
57	099	3353
58		6806
59	100	0257

the logarithms, with a decimal point prefixed to their first figure, are respectively the true or real logarithms of

1.255, 1.256, 1.257, 1.258, 1.259,

and since

$$\log.\ 12.55 = \log.\ (1.255 \times 10) = \log.\ 1.255 + \log.\ 10 = \log.\ 1.255 + 1,$$

and since similarly,

$$\log.\ 125.5 = \log.\ 1.255 = \log.\ 1.255 + 2, \&c.$$

we find by these properties of logarithms, the logarithms of 12.55, 125.5, 1255, &c. to be expressed by

$$1.0986437, \qquad 2.0986437, \qquad 3.0986437:$$

and similarly, we may, from the logarithms of 1.256, 1.257, &c. immediately assign, by prefixing the proper indices or characteristics, the logarithms of 12.56, 125.6, 12560, &c. 12.57, &c. 1257000, &c.

But if .0986437 be, as it is, the real logarithm of 1.255, the real logarithm of .1255 $\left(\text{since } .1255 = \frac{1.255}{10}\right)$ must equal

$$.0986437 - 1, \text{ or } - .9013562,$$

and the real logarithm of .01255 $\left(\text{since } .01255 = \frac{1.255}{100}\right)$ must equal

$$.0986437 - 2, \text{ or } - 1.9013562.$$

These negative quantities, then, are the real logarithms of the above decimal numbers, that is, the equations,

$$.12.55 = 10^{-.9013562}, \text{ or } = \frac{1}{10^{.9013562}}$$

$$.01255 = 10^{-1.9013562}, \text{ or } = \frac{1}{10^{1.9013562}}$$

$$.001255 = 10^{-2.9013562}, \text{ or } = \frac{1}{10^{2.9013562}},$$

$$\&c. = \&c.$$

are, within certain limits of exactness, true equations.

Now although, by means of the registered logarithms, the logarithms of decimal numbers may always be assigned by the preceding method, yet they are not immediately assigned: there intervenes, as an operation, the subtraction of the logarithm taken out of the Tables, either from 1, or 2, or 3, &c. In order to get rid of this subtraction, the Authors of the Rules or Precepts for the use of Logarithmic Tables have devised a notation for *negative* logarithms (which the logarithms of all proper fractions are) by which the number or the series of figures assigned by the Tables for the logarithm of any number may be retained. They have chosen to represent the logarithms of

.1255, .01255, .001255, &c.

neither by .0986437 − 1, .0986437 − 2, &c.

nor by the results − .9013562, − 1.9013562,

but by

$\bar{1}.0986437$, $\bar{2}.0986437$, &c.

and they apply a similar conventional notation to designate the logarithms of all other decimal numbers. The advantage of this notation is obvious: the same set of figures or of cyphers which the Tables assign to a number are to be forthwith used, whatever that number be, whether integer or a decimal fraction: thus, if in the Logarithmic Tables 8785218, under the column marked log., stands opposite the number 756, then, by the properties of logarithms (see pp. 210, &c.) and the peculiar notation,

Numbers.	Logarithms.	
75600	4.8785218	
7560	3.8785218	
756	2.8785218	
wrong 7.56	1.8785218	= 75.6
.756	0.8785218	= 7.56
.0756	$\bar{1}$.8785218	= .756
.00756	$\bar{2}$.8785218	= .0756
.00756	$\bar{3}$.8785218	= .00756

The first 5 logarithms are real numbers in which the figures and cyphers have, according to their order or arrangement, that significancy which they have in all ordinary arithmetical operations: the 3 last logarithms might be called *Artificial* Numbers, since their significancy cannot be inferred from analogy, but is altogether arbitrary or conventional.

Now, this being the case, we cannot, relatively to these latter logarithms, establish any rules for operating on them, that is, any rules for adding to them, or for multiplying and dividing them, except by reference to what they are made to stand for. In so doing we make a recurrence to a kind of *first principles.* In order then to add the logarithms,

$$\bar{1}.4329693 \text{ and } \bar{2}.6901961,$$

(which it is necessary to do in finding, by means of the Logarithmic Tables, the product of .271 and .049 of which the above quantities are respectively the logarithms) we must substitute the quantities they stand for: thus

$$\begin{array}{lrcl} (a) & \bar{1}.4329693 & = & .4329693 - 1 \\ (b) & \bar{2}.6901961 & = & .6901961 - 2 \\ \hline & & & 1.1231654 - 3. \end{array}$$

$$\text{Now } 1.1231654 \text{ means } 1 + .1231654;$$

$$\therefore 1.1231654 - 3 \text{ equals } .1231654 - 2,$$

which, according to the peculiar notation, may be thus written $\bar{2}.1231654$: which is the result that would be obtained by adding (*a*) and (*b*), and causing the unit *carried over* by the addition of 4 and 6 to destroy $\bar{1}$. Generally,

$$\begin{array}{rcl} \bar{n}.431 & = & .431 - n \\ \bar{m}.752 & = & .752 - m \\ \hline \therefore \bar{n} + \bar{m} + 1.183 & = & 1.183 - n - m \\ & = & .183 - (\bar{n} - 1) - m; \end{array}$$

but $.183 - (n - 1) - m$ may be written $\overline{n - 1} + \bar{m}.183$;

$\therefore$ in $\bar{n} + \bar{m} + 1.183$ the 1 is to be incorporated with the $\bar{n}$ or $\bar{m}$, and written thus,

$$\overline{m - 1} + \bar{m}.183, \text{ or } \overline{m - 1} + \bar{n}.183.$$

Suppose it necessary to subtract the preceding logarithms (which it would be were it required to divide, by means of logarithmic Tables .271 by .049.)

then since (*c*) $\bar{1}.4329693$ stands for $.4329693 - 1$,
and (*d*) $\bar{2}.6901961$ $.6901961 - 2$;
by subtraction, the right result is $-1 + .7227732 + 1$,
or $.7227732$

which is the same result as will follow by subtracting (*d*) from (*c*) in the common way and by considering -1 and $\bar{1}$ as the same.

In order to procure instances for the multiplication and division of such indices or characteristics, as $\bar{2}$, $\bar{3}$, &c. Suppose it were required to find, by the aid of the Logarithmic Tables, the value of $(.0756)^3$.

Now, by the properties of logarithms (see pp. 210, &c.)

$$\log.\ (.0756)^3 = 3 \log.\ 0756 = 3. \times \bar{2}.8785218$$

$$\begin{array}{lr} \text{Now } \bar{2}.8785218 \text{ stands for} & .8785218 - 2 \\ \text{(multiply by 3)} \ldots\ldots\ldots\ldots & 3 \\ \hline & 2.6355654 - 6 \end{array}$$

but $2.6355654 - 6$, is the same as $2 + .6355654 - 6$, which equals $.6355654$, which, according to the peculiar notation of logarithms (see p. 228.) may be thus noted $\bar{4}.6355654$: the same result as will be obtained by multiplying $\bar{2}.8785218$ by 3, considering $\bar{6}$ $(= 3 \times \bar{2}) + 2$ to be the same as $\bar{4}$.

Since $.6355654$ is the logarithm of 4.3208, $\bar{4}.6355654$ [which is the logarithm of $(.0756)^3$], is the logarithm of .00043208: consequently,

$$(.0756)^3 = .00043208.$$

When the index or characteristic is to be divided (which happens in finding the root of numbers by the method of logarithms) the operation is less direct: suppose it were required to find the value of $(.0756)^{\frac{1}{3}}$.

$$\text{Now } \log.\ (.0756)^{\frac{1}{3}} = \frac{1}{3} \log.\ 0756 = \frac{1}{3}\,(\bar{2}.8785218),$$

$$\begin{aligned}
\text{Now, } \bar{2}.8785218 &= .8785218 - 2 \\
&= 1 + .8785218 - 2 - 1 \\
&= 1 + .8785218 - 3 \\
\text{or } &= 1.8785218 - 3 \\
\therefore \frac{1}{3}(\bar{2}.8785218) &= \frac{1}{3}(1.8785218 - 3) \\
&= .6261739 - 1 \\
\text{or } & \bar{1}.6261739.
\end{aligned}$$

But it is evident we may at once obtain this result by changing $\bar{2}$, into $\bar{3}$, then by dividing by 3, and, in the division immediately succeeding to that of the index, carrying 1, as a quantity borrowed, to the next figure.

If the value of $(.0756)^{\frac{1}{5}}$ had been required, then, since it would be necessary to divide $\bar{2}.8785218$ by 5, we must make $\bar{2}$, $\bar{5}$, and *carry*, as quantity borrowed, 3 to the next figure: for

$$\begin{aligned}
\bar{2}.8785218 &= .8785218 - 2 \\
&= 3.8785218 - 5 \\
\therefore \frac{1}{5}(\bar{2}.8785218) &= \frac{1}{5}(\bar{5} + 3.8785218) \\
&= \bar{1}.7757043.
\end{aligned}$$

A few more instances, involving such characteristics as $\bar{2}$, $\bar{3}$, &c. are subjoined.

Required the sum of 3.6989700, $\bar{7}$.3467875, 1.4771213, $\bar{5}$.4313638,

by separate additions,

$$\begin{array}{r}
\bar{7}.3467875 \\
\bar{5}.4313638 \\
\hline
\overline{12}.7781513
\end{array}
\qquad\qquad
\begin{array}{r}
3.6989700 \\
1.4771213 \\
\hline
5.1760913 \\
\overline{12}.7781513 \\
\hline
\bar{7}.9542426
\end{array}$$

or by one operation,

$\bar{7}.3467875$
$\bar{5}.4313638$
3.6989700
1.4771213
$\overline{\bar{7}.9542426}$

Instances of Subtraction.

1st, $\bar{7}.9788107$ (= log. 00000095238)
subtrahend $\bar{3}.1549020$ (= log. 0014285)
$\bar{4}.8239087$ (= log. 00066666).

2d, $\bar{2}.2218487$ (= log. .016666)
subtrahend 4.6989700 (= log. .50000)
$\bar{7}.5228787$ (= log. .0000003333).

In the first of these instances the quotient (.00066666) arising from dividing .00000095238 by .0014285, is found by the method of logarithms: and in the second the quotient (.0000003333) arising from dividing .016666 by 50000.

Instances of Multiplication and Division.

Find the values of $(.05)^5$, of $(.0000625)^{\frac{1}{4}}$, and of $(.075218)^{\frac{2}{5}}$.

1st multiplicand $\bar{2}.6989700$ (the log. of .05)
multiplier 5
$\bar{7}.4948500$ (the log. of .0000003125)
$\therefore (.05)^5 = .0000003125$

2d, 4 | $\bar{5}.7958828$, the dividend (= log. 0000625)
$\bar{2}.9489707$ (= log. .088914);
$\therefore (.0000625)^{\frac{1}{4}} = .088914.$

3d, $\bar{1}.8763253$ (= log. .075218)
2
5 | $\bar{1}.7526506$ (see pp. 230, 231.)
$\bar{1}.9505301$ (= log. .89234);
$\therefore (.075218)^{\frac{2}{5}} = .89234.$

The logarithms of decimal fractions (see pp. 227, &c.) are truly and properly expressed by negative quantities: but since (see pp. 228.) they are not *commodiously* so expressed, a peculiar notation with negative *indices* or *characteristics* has been invented. Their meaning is to be derived not from analogy, but from the terms of that prescription that assigns them their meaning. We must refer, as we have seen (see pp. 229, &c.), to the same source for establishing the truth of rules for operating on such indices. But there is another contrivance for designating the logarithms of fractions, in which no negative indices are employed. This consists in *borrowing* 10, or 100, or 1000, &c. and by prefixing, to the decimal part of the logarithm, the difference between 10, or 100, or &c., and 1, 2, &c. Thus,

instead of $\bar{1}.8763253$, $\bar{7}.4948500$, &c.
the numbers 9.8763253, 3.4948500, &c.
or 99.8763253, 93.4948500, &c.

are written: but then, in these cases, to prevent ambiguities or to derive rules for operating on these *artificial* logarithms, it must be noted or understood that 10, 100, &c. is *borrowed*. For, .4948500, 1.4948500, really representing the logarithms of 3.125, 31.25, the logarithm 3.4948500 would naturally and analogously be that of 3125 ; but it is made, according to the notation we are now describing, to represent the logarithm of .0000003125. There will exist, therefore, in this, and in similar cases, an occasion of ambiguity which cannot occur in the notation with negative characteristics.

In order to establish any rules relatively to the last-mentioned method of noting the logarithms of fractions, we must, as in the case of negative characteristics, refer to the real quantities they are made to represent. For instance, 3.4945800 (which, without any convention expressed or implied, would designate the logarithm of 3125) is made to represent the logarithm .0000003125, it stands for $3.4945800 - 10$ ($= -6.5054199$): and this multiplied by 4 equals

$$4 \times 3.4945800 - 40$$
$$= 13.9783200 - 40,$$

which may be thus expressed,

13.9783200........40 being borrowed,
or 3.9783200........30 being borrowed;

whence we derive a rule for multiplication; which is, to multiply the logarithm (expressed by borrowing 10) by the multiplier (m for instance) and to reject the 10's from the characteristic: the number then borrowed is $m \times 10 -$ the 10's rejected.

Thus, if 5.9635786 stands for $.9635786 - 5$
6 or $.9635786 - (10-5)$
35.7814716 represents 6 times the logarithm of .00009195. Again, if 5.9635786 stands for .9635786 —(10—5), six times the logarithm equals 35.7814716, in which 60 is borrowed, or rejecting 30, 5.7814716 may represent it, 30 ($=60 - 30$) being borrowed.

In order to divide 9.7526506 by 5 (to take such an instance)

9.7526506 stands for $.7526506 - 1$,
or $9.7526506 - 10$;

$\therefore$ 49.7526506 is equivalent to $49.7526506 - 50$;

$$\therefore \frac{1}{5}(49.7526506) \text{ equivalent to } \frac{1}{5}(49.7526506) - 10,$$

or 9.9505301 to $9.9505301 - 10$;

but the left-hand set of figures is made to stand for the right: therefore we may divide a logarithm (expressed by the borrowing of 10) by changing the characteristic (c) into $\overline{m-1} \times 10 + c$, if m be the divisor, and then by dividing by m: the quotient is the real quotient, 10 being supposed to be borrowed, or is the real quotient $-$ 10; thus if 3.4948500 standing for $3.4948500 - 10$, be to be divided by 5, add 40 ($= \overline{5-1} \times 10$) to the characteristic making it 43: then

$$\frac{1}{5}(43.4948500) = 8.6989700.$$

There is then no difficulty in finding rules for operating with logarithms thus noted. But since (as we have already explained the matter in pp. 333, &c.) the logarithms, expressed by 10, or 100, or &c. being supposed to be borrowed, have a meaning different from their usual or natural import, there exists a cause of ambiguity and and some danger of confusion. The logarithms expressed by means of negative characteristics are free from these objections: they have indeed, like the others, a conventional meaning but they have only one meaning. The rules for operating with them are distinct: and the only objection against them, is, their typographical uncouthness.

Expressions for the Sine and Cosine of a Multiple Arc (see p. 46).

These expressions admit of no simple proofs; yet, we will proceed to give their proofs, principally from the necessity of the case, and, partly, on account of the importance of the formula on which they depend, and which is little known to English Students.

The object of the formula is to afford the means of computing ϕu (a function of u) when u is expressed by this equation,

$$u = x + y \cdot fu,$$

in which fu $(= U)$ denotes another function of u, not necessarily the same as ϕu *.

Since $u = x + y \cdot fu$, it is plain that u and consequently ϕu, depend on x and y, or, to use the language of Mathematicians, (which has been used on the Continent ever since the time of John Bernoulli) are *functions* of x and y; let $\phi u = V$; then, by taking the *partial* differential coefficients (see *Princ. Anal. Calc.* pp. 79, &c.) of V, we have

$$\frac{dV}{dx} = \frac{d(\phi u)}{du} \times \frac{du}{dx}, \quad \frac{dV}{dy} = \frac{d(\phi u)}{du} \times \frac{du}{dy};$$

$$\therefore \frac{dV}{dx} \times \frac{du}{dy} = \frac{dV}{dy} \times \frac{du}{dx} \ldots\ldots\ldots\ldots [a].$$

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* For instance, fu or U may in a particular case represent u^m, whilst ϕu represents sin. u, or tan. u (see *Anal. Calc.* pp. 42, &c.)

Again, take the partial differential coefficients of du, from the equation $u = x + y \cdot U$;

then,

$$\frac{du}{dx} = 1 + y \cdot \frac{dU}{du} \times \frac{du}{dx},$$

$$\frac{du}{dy} = U + y \cdot \frac{dU}{du} \times \frac{du}{dy}.$$

Eliminate, by means of these two equations, $\frac{dU}{du}$, and there results

$$\frac{du}{dy} - U \cdot \frac{du}{dx} = 0 \ldots\ldots\ldots\ldots\ldots\ldots [b].$$

Substitute in this last equation, the value of $\frac{du}{dy}$, as given in the equation [*a*], and

$$\frac{dV}{dy} - U \cdot \frac{dV}{dx} = 0 \ldots\ldots\ldots\ldots [c],$$

and from these two equations, the formulæ for u and for V ($= \phi u$) are to be derived. Thus,

$$\begin{aligned} u &= x + y \cdot fu \\ &= x + y \cdot f(x + y \cdot fu) \\ &= x + y \cdot X + y^2 \cdot A + y^3 \cdot B + \&c. \end{aligned}$$

$X = fx$, and A, B, &c. being supposed to be functions of x.

Hence,

$$\frac{du}{dx} = 1 + y \cdot \frac{dX}{dx} + y^2 \cdot \frac{dA}{dx} + y^3 \cdot \frac{dB}{dx} + \&c.$$

$$\frac{du}{dy} = X + 2y \cdot A + 3y^2 \cdot B + \&c.$$

Substitute these values in the equation [*b*], and there results

$$0 = X + 2y \cdot A + 3y^2 \cdot B + \&c.$$

$$- (X + y \cdot A + y^2 \cdot B) \times \left\{1 + y \cdot \frac{dX}{dx} + y^2 \frac{dA}{dx} + \&c.\right\}$$

Hence, comparing the coefficients of the terms involving like powers of y, we have

$$2A - A = X\frac{dX}{dx};\ \text{or},\ A = \tfrac{1}{2}\frac{d(X^2)}{dx}$$

$$2B = A.\frac{dX}{dx} + X.\frac{dA}{dx} = \frac{d(AX)}{dx}.$$

Now $AX = X^2.\dfrac{dX}{dx} = \tfrac{1}{3}\dfrac{d(X^3)}{dx};$

$$\therefore \frac{d(AX)}{dx} = \tfrac{1}{3}\frac{d\,d(X^3)}{dx^2} = \tfrac{1}{3}\frac{d^2(X^3)}{dx^2};$$

consequently, $B = \dfrac{1}{2\,.\,3}\,\dfrac{d^2(X^3)}{dx^2}$; and so on.

Hence,

$$u = x + yX + \frac{y^2}{2}.\frac{d(X^2)}{dx} + \frac{y^3}{2\,.\,3}.\frac{d^2(X^3)}{dx^2} + \&c.$$

and $U\left(=\dfrac{u-x}{y}\right) = X + \dfrac{y}{2}.\dfrac{d(X^2)}{dx} + \dfrac{y^2}{2\,.\,3}.\dfrac{d^2(X^3)}{dx^2} + \&c.$

ϕu is to be obtained precisely in the same way. For,

$$\phi u = \phi(x + y.fu) = \phi(x + yU);$$

$\therefore$, since the first term of the developement will be $\phi x\,(=P)$, and U has been expressed by a series of terms involving functions of x, assume

$$V, \text{ or } \phi u = P + y\,.\,Q + y^2.\,R + y^3.\,S, \&c.$$

P, Q, R, &c. being functions of x.

From this, find, as before, (p. 235, l. 28, 29.) the partial differential coefficients $\dfrac{dV}{dx}$, $\dfrac{dV}{dy}$ and substitute their values, with that of U, in the equation [c], and there results

$$0 =$$

$$Q + 2Ry + 3Sy^2 + \&c.$$

$$-\left(\frac{dP}{dx} + y.\frac{dQ}{dx} + y^2.\frac{dR}{dx} + \&c.\right) \times \left(X + \frac{y}{2}\frac{d(X^2)}{dx} + \&c.\right)$$

Hence, as before (p. 236.) multiplying the two series of the latter part, and comparing the terms affected with like powers of y,

$$Q = X\frac{dP}{dx}$$

$$\begin{aligned} 2R &= X\frac{dQ}{dx} + \frac{dP}{2\,.\,dx} \times \frac{d(X^2)}{dx} \\ &= X\frac{dQ}{dx} + \frac{Q}{2X} \times 2X.\frac{dX}{dx} \\ &= X\frac{dQ}{dx} + Q\frac{dX}{dx} \\ &= \frac{d(QX)}{dx}; \end{aligned}$$

$$\therefore R = \tfrac{1}{2}.\frac{d(QX)}{dx} = \tfrac{1}{2}\frac{d\left(X^2.\frac{dP}{dx}\right)}{dx}$$

and so on: Hence,

$$V, \text{ or } \phi u = P + yX\,.\,\frac{dP}{dx} + \frac{y^2}{2}\,\frac{d\left(X^2\frac{dP}{dx}\right)}{dx} + \&\text{c.}$$

which is the important formula that we proposed to investigate *.

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* See on this subject, Laplace, *Mem. Acad.* 1777, and his *Mec. Celeste*, pp. 170, &c. Lagrange, *Theorie des Fonctions Anal.* Edit. 1. pp. 101, &c. and his *Equa. Num.* Edit. 1. pp. 234, &c. and Cousin, *Astron. Physique*, pp. 15, &c.

The Student, who examines the preceding demonstration, will perceive how necessary it is, in complicated processes, to be possessed of a precise and commodious notation. In the text, the differential, in preference to the fluxionary, has been used; but even that may, with advantage, be, in this as in like cases, superseded by Arbogast's, in which the small Roman character D is used to denote the differential coefficient. With that character, instead of $\frac{d\left(X^2.\frac{dP}{dx}\right)}{dx}$ we should write $\mathrm{D}(X^2.\mathrm{D}P)$, and so on. See *Principles of Anal. Calc.* pp. 22, 43, &c. and Preface, pp. 28, &c.

We will now apply this theorem to express the cos. $m A$.

$$\text{By p. 48,}\quad 2 . \cos. m A = a^m + \frac{1}{a^m};$$

$$\text{if,}\quad 2 . \cos. A = a + \frac{1}{a}.$$

Hence, we must, from the preceding formula, find a^m and $\frac{1}{a^m}$ from this equation

$$a = 2 . \cos. A - \frac{1}{a};$$

for this end, compare it with $u = x + y . U$;

then $u = a$, $x = 2 \cos. A$, $y = -1$, and

$U \; (= f u) = \frac{1}{a}$: and the function (ϕu) of u, or of a, that is required, is a^m.

Hence, $P = \phi x = x^m = 2^m . \cos.^m A$

$$X = f x = \frac{1}{x} = \frac{1}{2 \cos. A}$$

$$\frac{d P}{d x} = m x^{m-1} = 2^{m-1} . m \cos.^{m-1} A;$$

$$\therefore\; X \frac{d P}{d x} = m \times 2^{m-2} . \cos.^{m-2} . A.$$

Again,

$$X^2 . \frac{d P}{d x} = m \times (2 \cos. A)^{m-3};$$

$$\therefore\; \frac{d\left(X^2 . \frac{d P}{d x}\right)}{d x} = m . (m-3)\, (2 \cos. A)^{m-4};$$

in like manner, $\frac{1}{a^m} = a^{-m}$ is to be obtained; in this case $\phi u = a^{-m}$, and $P = x^{-m}$,

and $\frac{d P}{d x} = - m x^{-m-1}$, whence,

$$X\frac{dP}{dx} = \frac{1}{2\,.\cos.\,A} \times -m(2\cos.\,A)^{-m-1} = -m.(2\cos.\,A)^{-m-2}$$

$$X^2\frac{dP}{dx} = -\,m.(2\cos.\,A)^{-m-3},$$

and $\therefore \dfrac{d\left(X^2.\dfrac{dP}{dx}\right)}{dx} = m.(m+3)(2\cos.\,A)^{-m-4}$, &c.

[*] Hence, $2.\cos.\,m\,A = a^m + \frac{1}{a^m} =$

$$(2.\cos.\,A)^m - m.(2\cos.\,A)^{m-2} + \frac{m.(m-3)}{2}(2\cos.\,A)^{m-4} + \&c.$$

$$+ (2\cos.\,A)^{-m} + m(2\cos.\,A)^{-m-2} + \frac{m(m+3)}{2}.(2\cos.\,A)^{-m-4} + \&c.$$

This expression is apparently composed of two series, one arranged according to the positive, the other according to the negative, powers of 2 cos. A. But, it is to be observed, the first series is to be continued, till the terms involve, in fact, negative powers of 2 cos. A, which, in the case of m being an integer, will be all destroyed by the terms of the second series.

For instance, if $m = 1$,

$$2\cos.\,A = 2\cos.\,A - (2\cos.\,A)^{-1} - (2.\cos.\,A)^{-3} + \&c.$$
$$+ (2\cos.\,A)^{-1} + (2.\cos.\,A)^{-3} + \&c.$$

Again, if $m = 2$,

$$2\cos.\,2A = (2\cos.\,A)^2 - 2.(2\cos.\,A)^0 - (2\cos.\,A)^{-2}\ \&c.$$
$$+ (2\cos.\,A)^{-2} + \&c.$$
$$= (2\cos.\,A)^2 - 2,$$

and, similarly, it will so happen, when other whole numbers are substituted for m.

Hence, in those cases in which m is represented by whole numbers, the value of 2 cos. $m\,A$ may be represented by the first series alone, that series being restricted from involving any negative powers of 2 cos. A: that is, if 2 cos. $A=p$,

---

[*] This is only one out of the many important applications that may be made of the formula which we have demonstrated.

$$2\,.\cos.\, m A = p^m - m p^{m-2} + \frac{m\,.(m-3)}{2}\, p^{m-4} - \&c.$$

The series is frequently written in a reverse order; since the general term is

$$\frac{m}{m-n} \times .\, \frac{(m-n)(m-n-1)\ldots\ldots(m-2n+1)}{1\,.\,2\ldots\ldots\ldots n} \times p^{m-2n};$$

when $m = 2n$, the series terminates, and the above term becomes $=2$. If we make the series begin from this term, then

$$2\,.\cos.\, m A = \pm\, 2\left\{1 - \frac{m^2 p^2}{1.2.2^2} + \frac{m^2(m^2-4)\,p^4}{1\,.\,2\,.\,3\,.\,4\,.\,2^4} - \&c.\right\}$$

the upper or lower sign taking place, as m is of the form $4s$, or $4s+2$.

If $m = 2n+1$, the preceding general term becomes mp, and then

$$2\cos.\, m A = \pm\left(m p - \frac{m.(m^2-1)}{1.2.3.2^2}\, p^3 + \frac{m(m^2-1)(m^2-9)}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,2^4}\, p^5 - \&c.\right)$$

the upper or lower sign taking place as m is of the form $4s+1$, or, $4s+3$.

From these expressions for the cosine of a multiple arc, several others may be deduced. Let s and c be the sine and cosine of an arc, then $c = \frac{p}{2}$: and

$$\cos.\, mA = \tfrac{1}{2}\left((2c)^m - m\,(2c)^{m-2} + \frac{m\,.(m-3)}{2}\,(2c)^{m-4} - \&c.\right) \; [a]$$

or, [m odd]

$$= \pm\left(m c - \frac{m\,.\,(m^2-1)}{1.2.3}\,.\, c^3 + \frac{m\,(m^2-1)\,(m^2-9)}{1.2.3\,.\,4.5}\, c^5 - \&c.\right) \; [b]$$

$$[m \text{ even}] = \pm\left(1 - \frac{m^2}{1.2}\, c^2 + \frac{m^2\,.\,(m^2-4)}{1\,.\,2\,.\,3\,.\,4}\, c^4 - \&c.\right)\ldots.[c].$$

Since the expression [b] is true whatever be the arc, let the arc instead of A, be $\frac{\pi}{2}-A$, then cos. $\left(\frac{\pi}{2}-A\right)$ = sin. $A = s$, and

cos. $m\left(\frac{\pi}{2}-A\right)$ = cos. $\left(\frac{m\pi}{2}-mA\right)$ = ± sin. mA, [+ if m

be of the form $4s+1$, − if of the form $4s+3$:] hence, in all cases, [m odd]

$$\text{sin. } mA = ms - \frac{m(m^2-1)}{1.2.3}s^3 + \frac{m.(m^2-1)(m^2-9)}{1.2.3.4.5}s^5 - \text{\&c. } [d]$$

Take the differential or fluxion of [a], then since [p. 106.]

$$d(\text{cos.} mA) = -\text{ sin. } mA.m.dA, \text{ and } 2^m.d(c^m) =$$

$$2^m \times mc^{m-1}.dc = -2^m.mc^{m-1}s.dA, \text{ \&c.}$$

∴ dividing by mdA, we have, since s enters each term, sin. mA =

$$s\left((2c)^{m-1}-(m-2)(2c)^{m-3}+\frac{(m-3)(m-4)}{2}(2c)^{m-5}-\text{\&c.}\right)^*[e].$$

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* The remark made in p. 244, l. 17, on formula [a], produced to the Author of the present Treatise an anonymous letter dated Trin. Coll. Dec. 12, 1817, communicating an ingenious demonstration of [e] (the formula for the sin. mA).

The following process for finding [a] contains, substantially, the principle of that demonstration.

By virtue of the formula of l. 8. p. 32,

$$\text{cos. }(n+1)a = 2\text{ cos. } na\text{ . cos. } a - \text{cos.}(n-1)a;$$

$$\therefore \text{ cos. }(n+1)a.x^n = 2\text{ cos. } na\text{ . cos. } a.x^n - \text{cos. }(n-1)a.x^n.$$

Let $n = 0, 1, 2, 3$, &c.; then collecting the severally resulting values

$$\text{cos. } a + \text{cos. } 2a.x + \text{cos. } 3a.x^2 + \text{\&c.} =$$

$$\text{cos. } a - x + 2.\text{cos. } a.x\ \{\text{cos. } a + \text{cos. } 2a.x + \text{cos. } 2a.x^2 + \text{\&c.}\}$$

$$-\ x^2\ \{\text{cos. } a + \text{cos. } 2a.x + \text{cos. } 3a.x^2 + \text{\&c.}\}$$

consequently,

$$\text{cos. } a + \text{cos. } 2a.x + \text{cos. } 3a.x^2 + \text{\&c.} + \text{cos. } na.x^{n-1} + \text{\&c.} =$$

If we perform the same operations on [*b*] and [*c*], we shall have [*m* odd] sin. *mA* =

$$\pm s\left(1-\frac{m^2-1}{2}c^2+\frac{(m^2-1)(m^2-9)}{2\,.\,3\,.\,4}c^4-\&c.\right)\ldots\ldots[f].$$

and [*m* even] sin. *m A* =

$$\frac{\cos.\,a-x}{1-2\cos.\,a\,.\,x+x^2}=\frac{\cos.\,a-x}{1+x^2}\times\frac{1}{1-2\cos.\,a.\frac{1}{1+x^2}}$$

$$1)\ =\{\cos.\,a-x.\}\left\{\begin{array}{l}\frac{1}{1+x^2}+2\cos.\,a.\frac{x}{(1+x^2)^2}+2^2\cos.^2 a\,.\frac{x^2}{(1+x^2)^3}+\&c.\\ +2^{n-4}\cos,^{n-4}a.\frac{x^{n-4}}{(1+x^2)^{n-3}}+2^{n-3}\cos.\,a^{n-3}\frac{x^{n-3}}{(1+x^2)^{n-2}}\\ +2^{n-2}\cos.^{n-2}a.\frac{x^{n-2}}{(1+x^2)^{n-1}}+2^{n-1}\cos.^{n-1}a.\frac{x^{n-1}}{(1+x^2)^n}+\&c.\end{array}\right\}$$

Equate the terms that on each side are affected with x^{n-1} : now of the terms within the brackets, the last written term, the last but two, the last but four, &c. each contain (when their denominators are evolved) one term and only one affected with x^{n-1} they will also be so affected when multiplied by cos. *a*, the first term of cos. $a-x$. The intermediate terms contain no such term : but they will contain such a term when multiplied by $-x$ the second term of cos. $a-x$: hence the coefficient of the term affected with x^{n-1} arising from the complete development of (*A*) is

$$\begin{array}{ll}2^{n-1}\cos.^n a & \\ -2^{n-3}\cos.^{n-2}a\,.\,(n-2) & -2^{n-2}\,.\cos.^{n-2}a\\ +2^{n-5}\,.\cos.^{n-4}a\,.\frac{(n-4)(n-3)}{1\,.\,2} & +2^{n-4}.\cos.^{n-4}a\,.\,(n-3)\\ -2^{n-7}\,.\cos.^{n-6}a\,.\frac{(n-6)(n-5)(n-4)}{1\,.\,2\,.\,3} & -2^{n-6}\cos.^{n-6}a.\frac{(n-5)}{1\,.\,2}(n-4)\\ +\&c. & +\&c.\end{array}$$

and hence cos. *n a* =

$$\frac{1}{2}\left\{\begin{array}{l}(2\cos.\,a)^n-n.(2\cos.\,a)^{n-2}+\frac{n(n-3)}{2}(2\cos.\,a)^{n-4}\\ -\frac{n.(n-4)(n-5)}{2\,.\,3}(2\cos.\,a)^{n-6}+\&c.\end{array}\right\}$$

$$\pm s\left(mc - \frac{m(m^2-4)}{2.3}c^3 + \frac{m(m^2-4)(m^2-16)}{2.3.4.5}c^5 - \&c.\right) [g].$$

If we perform a like operation on [*d*], then, since $d\,(\sin.\, mA) = m \cos.\, mA \,.\, dA$ [p. 106,], we shall have

$$\cos.\, mA = c\left(1 - \frac{m^2-1}{2}s^2 + \frac{(m^2-1)(m^2-9)}{2.3.4}s^4 - \&c.\right) \ldots.[h].$$

If in the equation [*c*] we put, instead of A, $\frac{\pi}{2} - A$, then

$$c = \cos.\left(\frac{\pi}{2} - A\right) = \sin.\, A = s,$$

$$\text{and } \cos.\, m\left(\frac{\pi}{2} - A\right) = \cos\left(\frac{m\pi}{2} - mA\right) = \pm \cos.\, mA$$

[+, if m is of form $4s$, −, if m is of form $4s + 2$.]

Hence, in both cases,

$$\cos.\, mA = 1 - \frac{m^2}{2}s^2 + \frac{m^2(m^2-4)}{2.3.4}s^4 - \&c. \ldots\ldots [i].$$

Take the differential or fluxion of this equation [*i*], and divide by $m\,dA$, then we have [m being even] $\sin.\, mA$

$$= c\left(ms - \frac{m(m^2-4)}{2.3}s^3 + \frac{m(m^2-4)(m^2-16)}{2.3.4.5}s^5 - \&c.\right) [k].$$

These formulæ for the sine and cosine of the multiple arc, (ten in number,) require not, as it has appeared, separate demonstrations, since the nine latter are easily derived from the first [*a*]; the attention then of the analyst ought principally to be directed to that. If in the formulæ [*a*], [*d*], [*e*], [*k*], we substitute for m, 2, 3, 4, 5, there will result, as particular instances, the forms designated by [c^{II}], [c^{III}], [c^{IV}], [c^{V}], [s^{III}], [s^{IV}], &c. in pages 45, 46, 47; and if in [*f*], [*h*], [*i*], we expound m by different numbers, we shall have

$$\left.\begin{array}{lll} m = 3 & \sin. 3A = & -s(1 - 4c^2) \\ m = 5 & \sin. 5A = & s(1 - 12c^2 + 16c^4) \end{array}\right\} \text{from } [f]$$

&c.

$$\left.\begin{array}{lll} m = 3 & \cos. 3A = & c(1 - 4s^2) \\ m = 5 & \cos. 5A = & c(1 - 12s^2 + 16s^4) \end{array}\right\} \text{from } [h]$$

&c.

$$\left.\begin{array}{lll} m = 4 & \cos. 4A = & 1 - 8s^2 + 8s^4 \\ m = 6 & \cos. 6A = & 1 - 18s^2 + 48s^4 - 32s^6 \end{array}\right\} \text{from } [i],$$

&c.

which particular forms were not deduced in the above-mentioned pages.

Series for the Sine and Cosine.

By the form $[d]$, p. 242,

$$\sin. mA = ms - \frac{m.(m^2-1)}{1.2.3}s^3 + \frac{m.(m^2-1)(m^2-9)}{2.3.4.5}s^5 - \&c.$$

Let A be very small, and m very large, and such, that $mA = x$, then $s = \sin. A = \sin. \frac{x}{m} = \frac{x}{m}$, nearly; and m^2-1, m^2-9, &c. $= m^2$, m^2, &c. nearly; $\therefore$ $\sin. mA$, or,

$$\sin. x = m.\frac{x}{m} - \frac{m.m^2}{2.3}.\left(\frac{x}{m}\right)^3 + \frac{m.m^2.m^2}{2.3.4.5}.\left(\frac{x}{m}\right)^5 - \&c.$$

$$= x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c. \ldots\ldots\ldots\ldots\ldots [v].$$

We shall have the same result if we take the series $[k]$ in which m is even; for $c = \cos. A = \cos. \frac{x}{m} = 1$ nearly, and m^2-4, m^2-16, &c. $= m^2$, m^2, &c.

If in the series $[h]$, or $[i]$, we make the same substitutions as we have already made, we shall have

$$\cos. mA, \text{ or } \cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c. \ldots\ldots [u].$$

Instead of computing sin. 1′ by the methods given in pp. 67 and 69, Calculators, in the construction of Trigonometrical Tables, have employed the preceding series [v] for the sin. x, and under this form,

$$\sin. \frac{m}{n} \cdot \frac{\pi}{2} = \frac{m}{n} \cdot \frac{\pi}{2} - \frac{m^3}{n^3} \cdot \frac{\pi^3}{2^3 . 2 . 3} + \&c.$$

putting $\frac{m}{n} \cdot \frac{\pi}{2}$ instead of x: they have also, availing themselves of previous computations,* taken $\pi = 3.14159\ 26535\ 89793$, and accordingly have been able to represent the above series, and the series [u] for the cosine, with numerical coefficients, after the following manner:

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* Dr. Horsley, in his *Elementary Treatise on Mechanics*, p. 153, says, that this is "taking things in a preposterous order;" and, undoubtedly, it would be so in a Treatise intended specially to explain the principles of the construction of the Trigonometrical Canon, but not in a Treatise giving rules for practically constructing it with as much ease and conciseness as possible.

* sin. $\frac{m}{n} \cdot 90^\circ =$

1.57079 63267 948966 $\frac{m}{n}$...[1]

− 0.64596 40975 062463 $\frac{m^3}{n^3}$...[2]

+.....7969 26262 461670 $\frac{m^5}{n^5}$...[3]

−......468 17541 353187 $\frac{m^7}{n^7}$...[4]

+........16 04411 847874 $\frac{m^9}{n^9}$

− ....35988 432352 $\frac{m^{11}}{n^{11}}$

+................569 217292 $\frac{m^{13}}{n^{13}}$

−..................6 688035 $\frac{m^{15}}{n^{15}}$

+.......................60669 $\frac{m^{17}}{n^{17}}$

−..........................438 $\frac{m^{19}}{n^{19}}$

cos. $\frac{m}{n}\, 90^\circ =$

1.00000 00000 000000

− 1.23370 05501 361698 $\frac{m^2}{n^2}$

+ 0.25366 95079 010480 $\frac{m^4}{n^4}$

−.....2086 34807 633530 $\frac{m^6}{n^6}$

+........91 92602 748394 $\frac{m^8}{n^8}$

−.........2 52020 423731 $\frac{m^{10}}{n^{10}}$

+.............4710 874779 $\frac{m^{12}}{n^{12}}$

−................63 866031 $\frac{m^{14}}{n^{14}}$

+....................656596 $\frac{m^{16}}{n^{16}}$

−.........................5294 $\frac{m^{18}}{n^{18}}$

+............................34 $\frac{m^{20}}{n^{20}}$

From this series not only the sin. 1′, but the sine of any arc may be computed; for instance, let $\frac{m}{n} = \frac{1}{10}$, then computing exactly as far as seven places,

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* Euler, *Introd. ad Anal. Inf.* p. 99. Callet's *Log*. 27, 28.

$$\left.\begin{array}{lr} \text{since } \frac{m}{n} = .1, & [1]=.157079632 \\ \text{since } \frac{m^5}{n^5} = .00001, & [3]=.000000796 \\ & \overline{.157080428} \\ \text{since } \frac{m^3}{n^3} = .001, & [2]=.000645964 \\ & \therefore \sin.\ 9^\circ = \overline{.156434464} \end{array}\right\}$$

The 4th term [4] and the remaining terms produce no significant figures in the 9th 8th, &c. places.

or, in nearest numbers, as far as 7 places, sin. 9° = .1564345. See p. 76, l. 6.

From the series for sin. $\frac{m}{n}$ 90°, we may, by assigning different values to $\frac{m}{n}$, deduce as many *formulæ of verification* as we please; for instance, suppose we wish to know whether sin. 20°, computed according to the methods of pages 70, 71, be rightly computed; put $\frac{m}{n}90^\circ = 20$; $\therefore \frac{m}{n} = \frac{2}{9}$, which value is accordingly to be substituted for $\frac{m}{n}$ in the several terms of the preceding series for sin. $\frac{m}{n} \times 90^\circ$.

The sines of arcs deduced by the preceding series and the formulæ of pages 70, 71, will be expressed in parts of the radius, and be, what are called, *natural sines;* but, computation is usually conducted by means of logarithmic sines, which latter may, by the aid of the common logarithmic Tables, if the log. sines are required to seven places only, be computed by taking the logarithms of the numbers that express the natural sines; and, in order to avoid the inconvenience of negative logarithms, (for if the radius = 1, the sines are all fractions and the logarithms consequently negative) the Trigonometrical Tables are constructed to a radius $= 10^{10}$, the logarithm of which = 10: so that, instead of $\bar{1}.6006997$, the logarithm of .3987491, which is the natural sine of 23° 30′ to a radius 1, $10 + (\bar{1}.6006997)$ or 9.6006997 is made to denote the logarithm.

But it is not absolutely necessary to compute the logarithmic from the natural sines; and, indeed, if the latter consist of more than 7 places, their logarithms cannot, from the Tables in common use, be obtained: on this account it becomes necessary to shew, by what means independent of Logarithmic Tables, or requiring the aid only of the Tables that are in ordinary use, logarithmic sines may be computed to any degree of exactness.

By the form [v], $\sin. x = x - \frac{x^3}{2\,.\,3} + \frac{x^5}{2\,.\,3\,.\,4\,.\,5} - \&c.$

$$= x\left(1 - \frac{x^2}{2\,.\,3} + \frac{x^4}{2\,.\,3\,.\,4\,.\,5} - \&c.\right)$$

Now, for the purpose of finding the composition of $1 - \frac{x^2}{2\,.\,3} + \&c.$ put $\sin. x = 0$, then (Table, p. 16. making $A = 0$) x may be 0, or π, or 2π, or 3π, &c. and x may also be either $-\pi$, or -2π, or, -3π, &c. hence, viewing the above series as an equation, 0, π, 2π, 3π, &c. are its roots, or if we put $\frac{1}{y}$ for x, and then reduce the equation, so that the term of the highest dimensions (y^n for instance) stands first, $\frac{1}{\pi}$, $\frac{1}{2\pi}$, $\frac{1}{3\pi}$, &c. $-\frac{1}{\pi}$, $-\frac{1}{2\pi}$, $-\frac{1}{3\pi}$, &c. are roots of $y^n - \frac{y^{n-2}}{2.3} + \&c.$ and consequently, for reasons like those stated in p. 57, $y - \frac{1}{\pi}$, $y - \frac{1}{2\pi}$, &c. $y + \frac{1}{\pi}$, $y + \frac{1}{2\pi}$, &c. are divisors of the equation: hence,

$$y^n - \frac{y^{n-2}}{2\,.\,3} + \&c. = \left(y - \frac{1}{\pi}\right)\left(y + \frac{1}{\pi}\right)\left(y - \frac{1}{2\pi}\right)\left(y + \frac{1}{2\pi}\right) \&c.$$

or $y^n\left(1 - \frac{1}{2.\,3\,y^2} + \&c.\right) = y\left(1 - \frac{1}{\pi y}\right) y\left(1 + \frac{1}{\pi y}\right) \&c.$

or (dividing by y^n),

$$1 - \frac{1}{2\,.\,3\,y^2} + \&c. = \left(1 - \frac{1}{\pi y}\right)\left(1 + \frac{1}{\pi y}\right) \&c.$$

$$\text{and } \therefore 1 - \frac{x^2}{2.3} + \&c. = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right) \&c.$$

Hence, $\sin. x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)$ &c.

or, $\left(\text{putting } x = \frac{m}{n} . \frac{\pi}{2}\right)$

$$\sin. \frac{m}{n} . \frac{\pi}{2} = \frac{m}{n} . \frac{\pi}{2}\left(1 - \frac{m^2}{4n^2}\right)\left(1 - \frac{m^2}{16n^2}\right) \&c.$$

and consequently, $\log. \sin. \frac{m}{n} . \frac{\pi}{2}$

$$= \log. \pi + \log. \frac{m}{2n} + \log.\left(1 - \frac{m^2}{4n^2}\right) + \log.\left(1 - \frac{m^2}{16n^2}\right) + \&c.$$

By a like decomposition we shall have

$$\cos. \frac{m}{n} . \frac{\pi}{2} = \left(1 - \frac{m^2}{n^2}\right)\left(1 - \frac{m^2}{9n^2}\right)\left(1 - \frac{m^2}{25n^2}\right) \&c.^*$$

$$\text{and } \log. \cos. \frac{m}{n} . \frac{\pi}{2} = \log.\left(1 - \frac{m^2}{n^2}\right) + \log.\left(1 - \frac{m^2}{9n^2}\right) + \&c.$$

---

* If in the expression for $\sin. \frac{m}{n} . \frac{\pi}{2}$, we put $n - m$ instead of m, then, since $\sin.\left(\frac{n-m}{n} . \frac{\pi}{2}\right) = \sin.\left(\frac{\pi}{2} - \frac{m\pi}{n.2}\right) = \cos. \frac{m}{n} . \frac{\pi}{2}$, we have $\cos. \frac{m}{n} . \frac{\pi}{2} = \frac{n-m}{n} . \frac{\pi}{2}\left(\frac{n+m}{2n}\right)\left(\frac{3n-m}{2n}\right)\left(\frac{3n+m}{4n}\right)\left(\frac{5n-m}{4n}\right)$ &c. if we equate this expression for $\cos. \frac{m}{n} . \frac{\pi}{2}$ with the former one, and divide each by the common factors, there results $\frac{1}{3.3.5.5.7.7, \&c.} = \frac{\pi}{2} \times \frac{1}{2.2.4.4.6.6, \&c.}$ whence $\frac{\pi}{2} = \frac{2.2.4.4.6.6, \&c.}{1.3.3.5.5.7.7, \&c.}$ which is Wallis's expression; and many other curious results, which are not, however, the proper objects of this Treatise, might be obtained.

The preceding series may be differently expressed: since *

$$\log.\left(1-\frac{m^2}{4n^2}\right)=\log.\left(\frac{4n^2-m^2}{4n^2}\right)$$

$$=\log.(2n+m)(2n-m)-2\log.2n$$

$$=\log.(2n+m)+\log.(2n-m)-2\log.2-2\log.n$$

$$\therefore\ \log.\sin.\frac{m}{n}.90^0=\log.\pi+\log.m-3\log.n-3\log.2$$

$$+\log.(2n+m)+\log.(2n-m)$$

$$+\log.\left(1-\frac{m^2}{4^2n^2}\right)+\log.\left(1-\frac{m^2}{6^2.n^2}\right)+\&c.$$

and by steps exactly similar we may obtain

$$\log.\cos.\frac{m}{n}.90^0=\log.(n-m)+\log.(n+m)-2\log.n$$

$$+\log.\left(1-\frac{m^2}{3^2n^2}\right)+\log.\left(1-\frac{m^2}{5^2n^2}\right)+\&c.$$

and since, (p. 214),

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* Log. $\left(1-\frac{m^2}{4n^2}\right)$, or log. $\left(1-\frac{m^2}{2^2n^2}\right)$ is not expanded, as the similar expressions log. $\left(1-\frac{m^2}{4^2n^2}\right)$, &c. are, and for this reason; if expanded, it would increase the coefficient of $\frac{m^{20}}{n^{20}}$ by $\frac{1}{2^{20}}$: now $\frac{1}{2^{20}}=\frac{1}{1048576}=.00000009$, &c. or the significant figures would come in the eighth place, whereas $\frac{1}{4^{20}}+\frac{1}{1099511627776}=.00000000000009$, &c. and the significant figures do not come in till the fourteenth place: if, therefore, log. $\left(1-\frac{m^2}{2^2n^2}\right)$ had been expanded, or the powers of $\frac{1}{2^2}$ retained in the computation, we must have computed a greater number of terms, (see succeeding series, p. 247,) in order to have had the series exact to fifteen places of decimals.

$$\log.\ \left(1-\frac{m^2}{4^2\,n^2}\right)=-\frac{1}{p}\left\{\frac{m^2}{4^2\,n^2}+\frac{m^4}{2.\,4^4\,n^4}+\frac{m^6}{3\,.\,4^6\,n^6}+\&c.\right\}$$

$$\log.\ \left(1-\frac{m^2}{6^2\,n^2}\right)=-\frac{1}{p}\left\{\frac{m^2}{6^2\,n^2}+\frac{m^4}{2\,.\,6^4 n^4}+\frac{m^6}{3\,.\,6^6\,n^6}+\&c.\right\}.$$

$$\log.\ \left(1-\frac{m^2}{8^2\,n^2}\right)=-\frac{1}{p}\left\{\frac{m^2}{8^2\,n^2}+\frac{m^4}{2\,.\,8^4\,n^4}+\frac{m^6}{3\,.\,8^6\,n^6}+\&c.\right\}$$

If we sum the coefficients of $\frac{m^2}{n^2}$, $\frac{m^4}{n^4}$, &c. taking the columns vertically, that is, if we find the arithmetical value of

$$.4342944,\ \&c.\ \left\{\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ \&c.\right\}\ \left(\text{coefficient of } \frac{m^2}{n^2}\right)$$

$$\tfrac{1}{2}\,.4342944,\ \&c.\ \left\{\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\ \&c.\right\}\ \left(\text{coefficient of } \frac{m^4}{n^4}\right)$$

$$\tfrac{1}{3}\,.4342944,\ \&c.\left\{\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\&c.\right\}\ \left(\text{coefficient of } \frac{m^6}{n^6}\right)$$

and add 10 the log. tabular radius, we shall have two Tables resembling those given in page 247.

$$\log.\ \sin.\ \frac{m}{n}.90^\circ =$$

$$\log. m + \log. (2n - m)$$

$$+ \log. (2n + m) - 3 \log. n$$

[a] $+\ 9.59405\ 98857\ 02190$

[1] $-\ 0.07002\ 28266\ 05901 \frac{m^2}{n^2}$

[2] $-\ \ldots\ldots 111\ 72664\ 41661 \frac{m^4}{n^4}$

[3] $-\ \ldots\ldots\ldots 3\ 92291\ 46453 \frac{m^6}{n^6}$

$$-\ \ldots\ldots\ldots\ldots 17292\ 70798 \frac{m^8}{n^8}$$

$$-\ \ldots\ldots\ldots\ldots\ldots 843\ 62986 \frac{m^{10}}{n^{10}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots 43\ 48715 \frac{m^{12}}{n^{12}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots 2\ 31931 \frac{m^{14}}{n^{14}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots 12659 \frac{m^{16}}{n^{16}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 702 \frac{m^{18}}{n^{18}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 39 \frac{m^{20}}{n^{20}}$$

$$\log.\ \cos.\ \frac{m}{n} 90^\circ =$$

$$\log. (n - m) + \log. (n + m) - 2 \log. n$$

$$+\ 10$$

$$-\ \ldots 0.10149\ 48593\ 41892 \frac{m^2}{n^2}$$

$$-\ \ldots\ldots\ldots 318\ 72940\ 65451 \frac{m^4}{n^4}$$

$$-\ \ldots\ldots\ldots\ldots 20\ 94858\ 00017 \frac{m^6}{n^6}$$

$$-\ \ldots\ldots\ldots\ldots 1\ 68483\ 48597 \frac{m^8}{n^8}$$

$$-\ \ldots\ldots\ldots\ldots\ldots 14801\ 93986 \frac{m^{10}}{n^{10}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots 1365\ 02272 \frac{m^{12}}{n^{12}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots 129\ 81715 \frac{m^{14}}{n^{14}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots 12\ 61471 \frac{m^{16}}{n^{16}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1\ 24567 \frac{m^{18}}{n^{18}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 12456 \frac{m^{20}}{n^{20}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1258 \frac{m^{22}}{n^{22}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 128 \frac{m^{24}}{n^{24}}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 13 \frac{m^{26}}{n^{26}}$$

From these series, may logarithmic sines and cosines, independently of the values of the natural sines, be computed to 15 places; and, this inconvenience is avoided; if the natural sines had been taken, consisting of more than 7 places, no Tables in common use would give their logarithms. The logarithms indeed of the numbers m, $2n-m$, $2n+m$, &c. are supposed to be taken to 15 places, and these can be had, since the numbers will not consist of more than 6 figures: for $\frac{m}{n}$ cannot exceed $\frac{1}{2}$*; therefore, since $n = 90.60.60 = 324000$, $n + m$, $2n+m$, &c. cannot exceed 1000000.

As an instance to the preceding formula, suppose the logarithmic sine of 9° to be required: here $m=1$, $n=10$.

∴ log. m or log. 1 = 0
log. $(2n - m)$ or log. 19 .... = 1.2787536009
log. $(2n + m)$ or log. 21 .... = 1.3222192947
[a] .... = 9.5940598857
12.1950327813 [d]
[1] .... .0007002282
[2] 1117
3 log. 10 .... = 3.
3.0007003399 [e]
∴ log. sin. 9°, that is, [d] − [e] .... = 9.1943324414

This is the log. sin. 9° to 10 places: and the decimal part is the logarithm of 15643446 the natural sine of 9°, found, p. 73, &c.

Rule, with its proof, for finding the logarithmic sine and tangent of very small arcs, (see pp. 96, &c. and Introduction to Taylor's *Logarithms*, p. 17, &c.)

For the sine; to the logarithm of the arc reduced into seconds with the decimal annexed, add the constant quantity 4.6855749, and from the sum subtract one-third of the Arithmetical complement of the log. cosine; the remainder will be the logarithmic sine of the given arc.

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* If $\frac{m}{n} > \frac{1}{2}$, the series for the cosine would be used for computing the sine, since sin. $(45° + A) =$ cos. $(45° - A)$: it is obvious, that the logarithms of m, $n+m$, &c. may be dispensed with entirely, by expanding log. $\left(1 - \frac{m^2}{2^2 n^2}\right)$; but then, to attain the same exactness, we must make the series consist of more terms. It is also plain, that instead of fifteen places in the numerical coefficients of the series, any number may be used. See Callet's *Logarithms*, p. 48.

For the tangent; to the log. arc and above constant quantity, add two-thirds of the Arithmetical complement of the log. cosine, the sum is the log. tangent of the given arc.

Proof of the Rule for the Sine.

By the series, p. 245, $\sin. x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c.$

$=$ (x being small)

$$x - \frac{x^3}{2.3} = x\left(1 - \frac{x^2}{2.3}\right) = x\left(1 - \frac{x^2}{2}\right)^{\frac{1}{3}} \text{ nearly,}$$

$$\text{hence, } \log. \sin. x = \log. x, + \tfrac{1}{3} \log. \left(1 - \frac{x^2}{2}\right)$$

$$= \log. x + \tfrac{1}{3} \log. \cos. x,$$

and introducing the tabular radius 10^{10}, the logarithm of which $= 10$, $\log. \sin. x = \log. x - \frac{1}{3}(10 - \log. \cos. x)$.

Now x is in parts of the radius; let n be its value reduced to seconds, then $2r \times 3.14159$, &c. : x :: $360.60.60$: n;

consequently,

$$\log. x = \log. n + \log. 2r + \log. 3.14159, \&c. - 2 \log. 36 - \log. 1000$$

$$= \log. n + 4.6855749,$$

and $\log. \sin. x = \log. n + 4.6855749 - \frac{1}{3}(10 - \log. \cos. x)$ which, since $10 - \log. \cos. x$ is the Arithmetical complement of the logarithmic cosines, proves the rule.

Proof of the Rule for the Tangent.

$$\tan. x = \frac{\sin. x}{\cos. x} = \frac{x - \frac{x^3}{2.3}}{1 - \frac{x^2}{2}}, \text{ nearly} = \frac{x\left(1 - \frac{x^2}{2}\right)^{\frac{1}{3}}}{1 - \frac{x^2}{2}} = \frac{x}{\left(1 - \frac{x^2}{2}\right)^{\frac{2}{3}}}$$

$\therefore$ log. tan. $x =$ log. $x - \frac{2}{3}$ log. cos. x;

or $=$ log. $x + \frac{2}{3}(10 -$ log. cos. $x)$ [introducing the tabular radius 10^{10}].

hence, as before, n being the number of seconds in the arc x, log. tan. $x =$ log. $n + 4.6855749 + \frac{2}{3}$ arith^l. comp^t. log. cos. x;

since cot. $x = \frac{r^2}{\text{tan. } x}$, log. cot. x, or log. tan. $(90^0 - x) =$

$$20 - (\text{log. } n + 4.6855749 + \tfrac{2}{3} \text{ arith}^l\text{. comp}^t\text{. log. cos. } x).$$

These are the proofs of the two rules for finding the sine and tangent from the arc; but, there are also two rules for the reverse operation of finding the arc from the log. sine and log. tangent. These are subjoined. See the Introduction to Taylor's *Logarithms*, p. 22.

Rule to find the Arc from the Sine.

To the given log. sine of a small arc add 5.3144251 and $\frac{1}{3}$ of the arithmetical complement of log. cosine; subtract 10 from the index of the sum, and you will have the logarithm of the number of seconds with the decimal fraction of the given arc.

Rule to find the Arc from the Tangent.

To the log. tangent add 5.3144251, and from the sum subtract $\frac{2}{3}$ of the arithmetical complement of log. cosine, and subtract 10 from the index, and you will have the logarithm of the number of seconds with the decimal fraction of the given arc.

Proof of the First Rule.

If $y =$ sine, and $x =$ arc, then $dx = \frac{dy}{\sqrt{(1 - y^2)}}$ ......(p. 106);

$$\therefore x = \int dy\,(1 - y^2)^{-\frac{1}{2}} = \int dy \left(1 + \frac{y^2}{2}\right) \text{ nearly, since } y \text{ is small,}$$

$$\therefore x = y + \frac{y^3}{2 \, . \, 3} \text{ (nearly)} = y\left(1 + \frac{y^2}{2 \, . \, 3}\right) = y\left(1 - \frac{y^2}{2}\right)^{-\frac{1}{3}}$$

$$\therefore \text{log.}\, x = \text{log.}\, y - \tfrac{1}{3}\, \text{log.} \left(1 - \frac{y^2}{2}\right) = \text{log.}\, y - \tfrac{1}{3}\, \text{log. cos.}\, x;$$

or to tabular radius, log. x = log. $y + \frac{1}{3}$ (10 − log. cos. x), but, p. 255, log. x = log. n + 4.6855749 = log. n + (10 − 5.3144251);

consequently,

log. n = log. y + 5.3144251 + $\frac{1}{3}$ arith^{l}. comp^{t}. log. cos. x − 10 which is the rule symbolically expressed.

Proof of the Second Rule.

$$dx = \frac{dy}{\surd(1-y^2)}, \text{ if } t \text{ be tan.} x,\ y = \frac{t}{\surd(1+t^2)} = t\,.(1+t^2)^{-\frac{1}{2}}$$

$$= t - \tfrac{1}{2} t^3 \text{ nearly, and } y^3 = \frac{t^3}{(1+t^2)^{\frac{3}{2}}} = t^3.(1+t^2)^{-\frac{3}{2}}$$

$$= t^3 - \tfrac{3}{2} t^5 \text{ nearly;}$$

hence, expanding, by the Binomial Theorem, the value of dx as far as two terms, and integrating, there results

$$x = y + \frac{y^3}{2\,.\,3}$$

$$= t - \frac{t^3}{2} + \frac{t^3.}{2\,.\,3} \text{ (nearly)}$$

$$= t - \frac{t^3}{3}$$

$$= t\,(1 + t^2)^{-\frac{1}{3}} \text{ (nearly).}$$

Hence, log. x = log. $t - \frac{1}{3}$ log. sec.^{2} x = log. $t - \frac{2}{3}$ log. $\dfrac{1}{\text{cos. } x}$, and supplying the Tabular radius, and putting for x its preceding value,

$$\text{log. } n = \text{log. } t + 5.3144251 - \tfrac{2}{3}\,(10 - \text{log. cos. } x) - 10$$

$$= \text{log. } t + 5.3144251 - \tfrac{2}{3} \text{ arith}^{l}.\ \text{comp}^{t}.\ \text{log. cos. } x - 10,$$

which is the rule symbolically expressed.

Examples to the Rules: with their Solutions by Sherwin's and Taylor's Tables.

Required the Arc whose Logarithmic Sine is 6.4976549.

By the Rule.

5.3144251
6.4976549
Arith^{l}. comp^{t}. = 0

11.8120800
∴ log. arc .... = 1.8120800
∴ arc .... = 64″.8754.

= 1′ 4″.8754

By Sherwin.

log. sin. 2′....... = 6.7647561
log. sin. 1′....... = 6.4637261

.3010300
given log. sin... = 6.4976549
log. sin. 1′....... = 6.4637261

339288

$$\text{arc} = 1' \frac{339288}{3010300} \times 60.$$

= 1′ 6″.76.

By Taylor.

log. sin. 1′ 5″ .... = 6.4984882
log. sin. 1 4........................ = 6.4917548

67334
given log. sin...................... = 6.4976549
log. sin. 1′ 4″ .... = 6.4917548

59001

$$\therefore \text{arc} = 1'\ 4'' + \frac{59001}{67334}$$

= 1′ 4″.876.

Required the Arc whose Logarithmic Tangent is 7.1644398.

By the Rule.

5.3144251
7.1644398

12.4788649

$10 + \frac{2}{3}$ ar co. log. cos. = 10.0000003

log. arc...... = 2.4788646
∴ arc..... = 301″.2067
= 5′ 1″.2067

By Sherwin.

log. tan. 6′...... = 7.2418778
log. tan. 5...... = 7.1626964

791814
given log tan..... = 7.1644398
log. tan..... = 7.1626964

17434

$$\therefore \text{arc} = 5' \frac{17434}{791814} \times 60''$$

= 5′ 1″.32.

By Taylor.

log. tan. 5′. 2″.................. = 7.1655821
log. tan. 5′. 1″.................. = 7.1641417

14404

given log. tan..................... = 7.1644398
log. tan. 5′. 1″.................. = 7.1641417

2981

$$\therefore \text{ arc} \ldots\ldots = 5'.\,1''\frac{2981}{14404}$$

$$= 5'\ 1''.2069.$$

Here, if we take, in the first Example, that to be the true result which is determined by the rule, it appears that the arc is not determined from its logarithmic sine by Sherwin's Tables exactly in seconds, nor by Taylor's Tables exactly in parts of seconds (see p. 96). In the second Example, the number of seconds in the result obtained by Sherwin's Tables is right: for, although the number of seconds is not necessarily right, it may, in certain cases, happen to be right: if, instead of 7.1644398, the proposed logarithmic tangent had been 7.2022871, the arc, by Sherwin, would have been 5′ 30″, and wrong in seconds, since the true arc is 5′ 28″.633.

Amongst the independent methods of ascertaining the true result, there is none more simple than that which is called, technically, the *differential* *. If a, a', a'', &c. be successive values of a quantity, a, differing by a constant interval 1, and if the 1st, 2d, 3d, &c. differences be d', d'', d''', &c. then any intermediate value (A) distant from a by the interval x is equal to

$$a + x.d' + x.\frac{x-1}{2}.d'' + x.\frac{x-1}{2}.\frac{x-2}{3}d''' + \&c.$$

This series, by a direct application, gives the logarithmic sine and tangent of a proposed arc: thus, suppose the logarithmic tangent of 5′ 1″ 12‴ 24$^{\text{iv}}$ to be required:

	d'	d''
a = log. tan. 5′ 0′ = 7.1626964		
a' = log. tan. 5′ 1″ = 7.1641417	14453	− 49
a'' = log. tan. 5′ 2″ = 7.1655821	14404	− 47
a''' = log. tan. 5′ 3‴ = 7.1670178	14357	

---

* See Wood, p. 242. Waring, 54.

Here, stopping at the 2d difference d'' and taking 48 for its mean value, and, for greater simplicity, making the series to begin from a', in which case

$$x = \frac{12 \times 60 + 24}{60.60} = \frac{62}{300}, \text{ we have}$$

$$\text{log. tan. arc required} = a' + x d' + x \cdot \frac{x-1}{2} d''$$

$$= 7.1641417 + .0014404 \times \frac{62}{300} + .0000004 = 7.1644398$$

as it ought to be. See the former Example, p. 258.

The arc cannot by a direct process be found, by means of the preceding series, from the logarithmic sine or tangent: but, thus it may be found:

Since x is small,

$$a' + x d' + x\frac{x-1}{2} d'' \; (=A) = a' + x d' - \frac{x}{2} d'' \text{ nearly.}$$

Hence, $x = \dfrac{A - a'}{d' - \dfrac{d''}{2}}$; thus, if $A = 7.1644398$;

then (by the Tables) the logarithmic tangents next less and greater are 7.1641417, the log. tan. 5′ 1″, and, 7.1655821 the log. tan. 5′ 2″: hence, taking the differences as before,

$$x = \frac{7.1644398 - 7.1641417}{.0014404 + .0000024} = .2067,$$

and consequently the arc $= 5' \; 1''.2067$ as before, see Example, p. 258. This method, although it does not determine the logarithmic sines and tangents of small arcs so concisely and conveniently as Dr. Maskelyne's Rule, is of very extensive use, and especially in Astronomy. It may be considered as the foundation of the common Rule given in books of Logarithms, for finding the logarithmic sines, tangents, &c. of arcs that contain a fractional part of a minute or second: for, in mean arcs, the differences d'', d''', &c. are nothing or very small, and consequently

$$A = a + x d'.$$

Thus, to find by Sherwin's Tables the log. tan. 44°. 29′ 30″.

log. tan. 44° 28′ = 9.9919143	d'	d''
log. tan. 44° 29′ = 9.9921670	2527	0
log. tan. 44° 30′ = 9.9924197	2527	

∴ since $x = \frac{1}{3}$, log. tan. 44° 29′ 30″ = 9.9921670 + $\frac{1}{2}$ (.0002527) = 9.9922933. See p. 96. Also Sherwin, pp. 23, 24, &c. Hutton, p. 149, &c.

Demonstration of the Formula for Computing the Approximate Reduction to the Horizon. See p. 186.

Let A be the observed angle, x the correction, that is, let $A + x$ be the angle required, H and h the heights,

$$\text{then } \cos.(A + x) = \frac{\cos. A - \sin. H . \sin. h}{\cos. H . \cos. h} \text{ (p. 139).}$$

But, p. 245, Appendix,

$$\sin. H = H - \frac{H^3}{2 . 3} + \text{\&c., when } H \text{ is small, } = H \text{ nearly;}$$

$$\text{similarly, } \sin. h = h$$

$$\cos. H = 1 - \frac{H^2}{2} + \text{\&c.} = 1 - \frac{H^2}{2} \text{ nearly; similarly, } \cos. h = 1 - \frac{h^2}{2}.$$

$$\text{Hence, } \cos.(A + x) = \frac{\cos. A - Hh}{\left(1 - \frac{H^2}{2}\right)\left(1 - \frac{h^2}{2}\right)} = \frac{\cos. A - Hh}{1 - \left(\frac{H^2}{2} + \frac{h^2}{2}\right)} =$$

$$(\cos. A - Hh)\left[1 - \tfrac{1}{2}(H^2 + h^2)\right]^{-1}$$

or, $\cos. A . \cos. x - \sin. A . \sin. x = (\cos. A - Hh)\left[1 + \frac{1}{2}(H^2 + h^2)\right]$ nearly,

but, since x is very small, $\cos. x = 1$, and $\sin. x = x$;

$$\therefore \cos. A - \sin. A . x = \cos. A - Hh + \tfrac{1}{2} \cos. A\,(H^2 + h^2)$$

$$\text{and } x = \frac{Hh - \frac{1}{2} \cos. A\,(H^2 + h^2)}{\sin. A};$$

or x may be differently expressed; thus,

$$\text{make } p = \tfrac{1}{2}(H + h) \qquad \therefore p^2 - q^2 = Hh$$

$$q = \tfrac{1}{2}(H - h) \qquad p^2 + q^2 = \tfrac{1}{2}(H^2 + h^2),$$

$$\text{and } x = \frac{p^2 - q^2}{\sin. A} - \frac{(p^2 + q^2) \cos. A}{\sin. A}$$

$$= p^2 . \frac{1 - \cos. A}{\sin. A} - q^2 . \frac{1 + \cos. A}{\sin. A},$$

$$\text{(p. 33.)} = p^2 . \tan. \frac{A}{2} - q^2 . \cot. \frac{A}{2} .$$

This is the expression given by M. Legendre, *Mem. Acad.* 1787, p. 354; also in his *Trig*. edit. 6, p. 413. It is nearly equal to the first term of the series given by Delambre, *Connoissance des Tems*, 1793, and in the first vol. of *Measure of an Arc of the Meridian*, p. 140; by Suanberg, p. 38; and by Dr. Maskelyne, *Phil. Trans.* 1797, p. 451.

If $n = \tan. \frac{A}{2} . \sin.^2 \frac{1}{2} (H+h) - \cot. \frac{A}{2} . \sin.^2 \frac{1}{2} (H - h)$

then, the series deduced by the three latter Mathematicians, is

$$x = n \sec. H. \sec. h - \frac{n^2}{2} \sec.^2 H. \sec.^2 h. \cot. A. \sin. 1''$$

$$+ \tfrac{1}{2} n^3 . \sec.^3 H. \sec.^3 h \left(\tfrac{1}{3} + \cot.^2 A\right) . \sin.^2 1'' + \&c.$$

Here the first term, since $\sin. \frac{1}{2} (H+h)$ is nearly $= \frac{1}{2} (H + h)$, agrees with Legendre's expression.

Demonstration of Legendre's Theorem, See p. 191.

Let r be radius of sphere; a, b, c, the sides of the spherical triangle; then the sides of a similar triangle on a sphere whose radius is 1, are $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$; let also A be the spherical angle opposite to a, and A' the angle of a rectilinear triangle, the sides of which are a, b, c; then, since $\frac{1}{\sin. \frac{b}{r} \sin. \frac{c}{r}} = \left(\sin. \frac{b}{r} \sin. \frac{c}{r}\right)^{-1}$

we have, by Prop. 18, p. 159,

$$\frac{1 + \cos. A}{2} = \left(\sin. \frac{S}{r} . \sin. \frac{S - a}{r}\right) \left(\sin. \frac{b}{r} . \sin. \frac{c}{r}\right)^{-1}$$

$$= \left(\frac{S}{r} - \frac{S^3}{2 . 3 r^3}\right) \left(\frac{S - a}{r} - \frac{(S - a)^3}{r^3}\right) \times \left(1 - \frac{b^2}{2 . 3 r^2} - \frac{c^2}{2 . 3 r^2}\right)^{-1}$$

$$\times \frac{r^2}{bc}$$

$$= \frac{S\,(S-a)}{r^2} \times \left(1 - \frac{S^2}{2\,.\,3r^2} - \frac{(S-a)^2}{2\,.\,3\,r^2}\right) \times \left(1 + \frac{b^2+c^2}{2\,.\,3\,r^2}\right) \times \frac{r^2}{b\,c}$$

$$= \frac{S\,.\,(S-a)}{b\,c} - \frac{S\,.\,(S-a)}{b\,c}\left(\frac{S^2+(S-a)^2-b^2-c^2}{2\,.\,3\,.\,r^2}\right)$$

neglecting the terms that involve, in the denominator, higher powers than r^2; but,

$$S^2 + (S-a)^2 - b^2 - c^2 = \frac{(b+c+a)^2}{4} + \frac{(b+c-a)^2}{4} - b^2 - c^2$$

$$= \frac{a^2+2\,b\,c-b^2-c^2}{2} = \frac{a^2-(b-c)^2}{2}$$

$$= \frac{(a+b-c)\,(a-b+c)}{2} = 2\,.\,(S-c)\,(S-b).$$

Hence, $\dfrac{1 + \text{cos.}\,A}{2} = \dfrac{S\,.\,(S-a)}{b\,c} - \dfrac{2\,S\,(S-a)\,(S-b)\,(S-c)}{b\,c \times 6\,r^2}$

(Prop. 2, p. 25.) $= \dfrac{1 + \text{cos.}\,A'}{2} - \dfrac{b\,c}{2 \times 6\,r^2}\,\text{sin.}^2\,A'$;

$$\therefore\ \text{cos.}\,A = \text{cos.}\,A' - \frac{b\,c}{6\,r^2}\ \text{sin.}^2\,A';$$

but, cos. $A = $ cos. $A' -$ sin. $A'.x$, if $A = A' + x$, and x be very small;

$$\therefore\ x = \frac{b\,c}{6r^2}\,.\,\text{sin.}\,A' = \frac{1}{3\,r^2} \times \frac{b\,c\,.\,\text{sin.}\,A'}{2} = \frac{\text{Area}}{3\,r^2},$$

and $A' = A - \dfrac{\text{Area}}{3\,r^2}$; and, in like manner, if B, C, B', C' be the other angles,

$$B' = B - \frac{\text{Area}}{3\,r^2}$$

$$C' = C - \frac{\text{Area}}{3\,r^2};$$

and $A' + B' + C' (= 180^\circ) = A + B + C - \frac{\text{Area}}{r^2}$;

or, $\frac{\text{Area}}{r^2} = A + B + C - 180$: and thus is proved to be true the theorem mentioned, p. 191.

THE END.